Invariant measures of the 2D Euler and Vlasov equations

Freddy Bouchet[1,2,3,*] and Marianne Corvellec[1]

September 16, 2010

Affiliations:

[1] : INLN, CNRS, UNS, 1361 route des Lucioles, 06560 Valbonne, France

[2] : CNLS, LANL, MS B258 Los Alamos National Laboratory, PO Box 1663, Los Alamos, NM 87545, United States

[3] : Laboratoire de Physique, ENS de Lyon, CNRS, Université de Lyon, 46 allée d'Italie, 69364 Lyon cedex 07, France

Email: *Freddy.Bouchet@ens-lyon.fr, Marianne.Corvellec@ens-lyon.fr

Keywords: 2D Euler equations, Vlasov equations, invariant measures, invariant measures for partial differential equations, 2D turbulence, geophysical turbulence, equilibrium statistical mechanics, long-range interactions.

Abstract

We discuss invariant measures of partial differential equations such as the 2D Euler or Vlasov equations. For the 2D Euler equations, starting from the Liouville theorem, valid for N-dimensional approximations of the dynamics, we define the microcanonical measure as a limit measure where N goes to infinity. When only the energy and enstrophy invariants are taken into account, we give an explicit computation to prove the following result: the microcanonical measure is actually a Young measure corresponding to the maximization of a mean-field entropy. We explain why this result remains true for more general microcanonical measures, when all the dynamical invariants are taken into account. We give an explicit proof that these microcanonical measures are invariant measures for the dynamics of the 2D Euler equations. We describe a more general set of invariant measures, and discuss briefly their stability and their consequence for the ergodicity of the 2D Euler equations. The extension of these results to the Vlasov equations is also discussed, together with a proof of the uniqueness of statistical equilibria, for Vlasov equations with repulsive convex potentials.

Even if we consider, in this paper, invariant measures only for Hamiltonian equations, with no fluxes of conserved quantities, we think this work is an important step towards the description of non-equilibrium invariant measures with fluxes.

*Author to whom any correspondence should be addressed.

1 Introduction

In a complex and chaotic dynamical system, such as a Hamiltonian system with a large number of degrees of freedom, or a non-equilibrium steady state where dissipation balances forcing on average, the knowledge of a non-trivial invariant measure is equivalent to the knowledge of the statistical properties of all physical quantities. It is thus an essential concept. In a turbulent problem, the knowledge of an invariant measure gives access to the stationary probability distribution function of all physical quantities, and gives a solution to the usual hierarchy of the n-point correlator of the velocity field, among other things. A series of very interesting recent works have proved the existence of invariant measures and described some of their properties, for instance in stochastic systems forced by noises [44, 43, 50, 16]. However, only in very few instances of complex systems, is an invariant measure explicitly known. Finite-dimensional Hamiltonian systems are among these exceptions: thanks to the Liouville theorem, a uniform measure on a constant-energy shell of phase space is invariant (microcanonical measure); the canonical Gibbs measures are other explicit examples of invariant measures. This essential remark is at the base of equilibrium statistical mechanics.

For Hamiltonian partial differential equations, the situation is more complex. Indeed, the dimension of the system is then infinite. Thus, the meaning of phase space volume and the proper normalization of a uniform measure over a constant-energy shell of phase space are not clear notions. Microcanonical or Gibbs measures then have to be built carefully and their properties have to be checked. There are a few examples, where a Gibbs-type invariant measure has been proved to exist (see for instance [14, 15] in the case of the nonlinear Schrödinger equation, see also [45, 51]). As in [14, 15], such a proof usually involves the study of approximations of the partial differential equations with finite dimension N, and of limits of ensembles of measures when N goes to infinity. That the limit is actually an invariant measure of the initial partial differential equation completes the proof. We consider in this paper the construction of microcanonical and Gibbs measures for the 2D Euler equations and for the Vlasov equations. We also consider other sets of invariant measures for these equations, and investigate their dynamical stability.

The flow of a perfect fluid is described by the Euler equations, one of the oldest equations in mathematical physics [30]. More than two and a half centuries after their discovery by Euler, these equations still offer great challenges to both mathematicians and physicists [30]. Two-dimensional flows and the two-dimensional Euler equations are mathematically much simpler than their three-dimensional counterparts, but still present some very interesting unsolved problems. The strong analogies between the 2D Euler equations and the Vlasov equations have been observed at least since the '50s: they are both nonlinear transport equations, the non-linearity being due to non-local and non-integrable (at long distance) interactions (long-range interactions) [1]. We hope that this work may be very useful in the large number of systems with long range interactions that have been studied recently [9, 26, 18, 10, 20].

One of the main physical phenomena arising from these two equations is the self-organization into large-scale coherent structures: large-scale particle clusters for the Vlasov equations with attractive potential, large-scale particle clouds (whose density profiles depend on both the dynamical invariants and the external confining potential) for the Vlasov equations with repulsive potential, or monopoles, dipoles, and parallel

[1] Whereas in the 2D Euler and Vlasov equations, the long-range interactions and the associated non-additivity are essential for thermodynamical properties of the system (leading possibly to statistical ensemble inequivalence), they are not essential in the following discussion dealing with invariant measures and dynamics. For instance all the discussion of this paper can be easily adapted to the Quasi-Geostrophic model for which the interaction decays exponentially for distance much larger than the Rossby deformation radius.

flows for the 2D Euler equations. Such large-scale structures are analogous to geophysical cyclones, anticyclones, and jets in the oceans and atmospheres [13]. This analogy, understood thanks to the strong theoretical similarities between the 2D Euler equations on one hand, and the quasi-geostrophic or shallow-water models on the other hand, is one of the main motivations for the study of the 2D Euler equations. The 2D Euler equations also describe experimental flows, such as the transverse dynamics of electron plasma columns [65], the large–Reynolds-number approximation of the dynamics of fluids when three-dimensional motion is constrained by a strong transverse field (rotation, transverse magnetic field in a liquid metal, etc. [67]), or the dynamics of fluids in very thin geometries [58].

As first guessed by Onsager [57], such a self-organization is naturally explained by equilibrium statistical mechanics: the infinite number of degrees of freedom involved should have a macroscopic behavior that corresponds to an average over a microcanonical measure. Because of the long-range interactions between vortices, such an equilibrium is not uniform and depends strongly on the boundary conditions. This explains the formation of vortices and jets. In his work [57], Onsager circumvented the difficulty of infinite dimensions discussed above, by studying the point-vortex model, which is a finite-dimensional Hamiltonian system, describing the dynamics of singular vortices. This model is actually a special class of solutions to the 2D Euler equations (please see [19, 22, 27] and references therein for recent developments of the kinetic theory of point-vortices). The equilibrium statistical mechanics of the point-vortex model has a long and very interesting history, with wonderful pieces of mathematical achievements [57, 38, 17, 40, 28, 19, 31, 32].

The study of the equilibrium statistical mechanics of the 2D Euler equations or of the Vlasov equations with usual smooth initial conditions implies considering the real infinite-dimension nature of the problem. An equilibrium statistical mechanics was proposed for these equations, based on the maximization of a Boltzmann–Gibbs entropy constrained by the conservation of the invariants, by Lynden-Bell for the Vlasov case [48], and by Robert, Sommeria, and Miller for the 2D Euler case [59, 54, 68] (see also [24] for a discussion on the analogies between both systems). Such an approach is basically a mean-field one: the use of the Boltzmann–Gibbs entropy for the entropy (or mean-field entropy) can be justified with the classical Boltzmann counting argument, which is meaningful only if correlations between the vorticity values (or values of the one-particle distribution in the Vlasov case) at different points can be neglected.

Although this mean-field approach was a phenomenological assumption in the first papers [48, 59, 54, 60], it can be proved to be exact in systems with long-range interactions. For instance, physicists discussed the validity of the mean-field approach for the point-vortex model in the '80s, arguing that a Kramer-Moyal expansion when N goes to infinity is self-consistent. During the '90s, mathematicians and mathematical physicists proved that the mean-field approach is correct for the point-vortex model, using different tools [17, 31, 40]. For the 2D Euler and Vlasov equations, following the work of Bourgain for the nonlinear Schrödinger equation, such a proof should involve three steps:

1. Proving Liouville theorems for finite-dimensional approximations of the dynamical equations, and defining microcanonical measures for those,

2. Studying the infinite-N limit for this set of measures and proving that the limit is actually described by the maximization of a mean field entropy,

3. Proving that the limit measure is a dynamically invariant measure of the 2D Euler (or Vlasov) equations.

For Galerkin truncations of the 2D Euler equations, e.g., using Fourier mode decomposition, Point 1 above is a classical result [46]; actually, the Euler equations verify a detailed Liouville theorem [42]. More

recently, Robert [61] proved that a much larger class of approximations of the 2D Euler equations, obtained by L_2 projections (e.g., finite-element approximations), verify a Liouville theorem. This last point is very important, because measures based on spatial truncations are much more natural than those based on Fourier mode decompositions for systems with long-range interactions. This makes the proof of Point 2 much easier. Indeed, Michel and Robert [52] proved large-deviation results for ensembles of Young measures where the mean-field entropy appears as the opposite of the large-deviation rate function. A similar large-deviation result, justifying also the maximization of a mean-field entropy, was obtained from sets of measures based on spatial discretization [6, 7], but without any reference to the dynamics. These two results are essential steps, but give only a partial answer to Point 2, as the relation to dynamics is missing, making any further step towards Point 3 more difficult. As noted in [61], another difficulty is that Fourier or L_2 projections conserve only the quadratic invariants. Then, finite-dimensional measures that are based on the Liouville theorem and that take into account further invariants, which is the relevant procedure, are not invariant for the finite-dimensional approximation of the dynamical system. This makes Point 3 out of reach. For instance, the derivation of Point 3 for the nonlinear Schrödinger equation by Bourgain [14, 15] relies deeply on the fact that this equation has only quadratic and linear invariants, and that finite-dimensional approximate measures are invariants for the finite-dimensional dynamical approximations.

From this discussion, we conclude that although the validity of the mean-field approximation and the validity of the maximization of a mean-field entropy for some sets of natural measures have been proved [53, 6, 7], a clear proof of the relation of those microcanonical measures with dynamics is still missing. Another important question is to know whether all measures built from any reasonable finite-dimensional truncation would lead to a unique limit measure in the limit of infinite dimension.

As regards the relation between microcanonical (or canonical) measures and dynamics, we argue in this paper that it is not necessary to follow the classical program described above (points 1, 2, 3). More precisely, we argue that there is no logical need for the finite-dimensional approximate measures to be invariant measures of the finite-dimensional approximations of the dynamical system. We can indeed rely on the large-deviation results [53, 6] and verify only *a posteriori* that the limit measure is an invariant measure of the 2D Euler equations. This is what we do in this paper, by studying directly the evolution equation for measures of the 2D Euler equations and studying ensembles of invariant measures. The discussion relies on Young measures, i.e., product measures for which the vorticity values at different points in space are independent random variables. The importance of Young measures in the context of equilibrium statistical mechanics was stressed in [52, 61]. One of the main motivations for the present work is to give a dynamical meaning to Young measures, that we guess will be essential for further developments.

We write the evolution equation for the characteristic functional of the vorticity field, for the 2D Euler dynamics. Applying these equations to Young measures, we can study the class of Young measures that are invariant. We then note that microcanonical (or canonical) measures, which are actually Young measures, are dynamically invariant. The class of invariant Young measures is however larger; we explain that this property is related to a breaking of ergodicity for the 2D Euler equations. The dynamical approach of this paper also allows to discuss the dynamical stability of invariant measures, and particularly of invariant Young measures. The possibility to discuss dynamics is one of the main motivations for this work, as will be further discussed in the conclusion.

In this paper, we also give a partial answer to the second question above: would any truncation lead to the same limit measure? For this we consider the case of the energy-enstrophy measure (which takes into

account only the energy and enstrophy as dynamical invariants). With direct explicit computations of the expectations for the Fourier N-dimensional approximate measures, we prove that the limit measure is the same as the one obtained by maximizing the mean-field entropy (obtained as a large-deviation functional of a Young measure or from spatially discretized systems).

Energy-enstrophy measures were first investigated by Kraichnan [41], in the canonical ensemble, and without considering the limit of an infinite number of degrees of freedom. In this paper, for the first time, we derive explicitly the microcanonical measure, which is the relevant physical one, for it takes into account the constraints explicitly. This derivation is based on expressing the constraint as a complex integral of an auxiliary variable, and analyzing precisely the integral asymptotics through the saddle-point approximation. This allows to actually take the limit of an infinite number of degrees of freedom with a fixed value of the invariants. This is a good opportunity to discuss again the Kraichnan theory: although the microcanonical measure leads to the same qualitative prediction as Kraichnan's, i.e., the condensation of energy in the largest-scale mode, we prove that microcanonical and canonical ensembles are not equivalent. More precisely, this is a situation of partial ensemble equivalence (see [29] for a definition).

In section 2, we introduce the 2D Euler equations and their invariants, we define the microcanonical measure, and we give a pedagogic heuristic account of the meaning of the mean-field approximation and of the resulting statistical equilibrium distributions.

In section 3, we perform the computations for the energy-enstrophy measure, both directly from its definition through Fourier modes and from the mean-field variational problem. We show that the results are equivalent and discuss ensemble inequivalence as well as the Kraichnan theory.

Section 4 is devoted to the dynamics of measures for the 2D Euler equations. We characterize the set of invariant Young measures and their stability. In section 5, we briefly explain the generalization of these results to the Vlasov equations. We also prove the uniqueness of statistical equilibria in the case of convex repulsive interaction potentials.

The relation of this work to non-equilibrium problems is discussed in section 6.

2 Equilibrium statistical mechanics of two-dimensional flows

2.1 2D Euler equations

Defining the vorticity as $\omega = (\nabla \times \mathbf{v}) \cdot \mathbf{e}_z$, the 2D Euler equations take the simple form of a conservation law for the vorticity. They read

$$\frac{\partial \omega}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \omega = 0 \ ; \quad \mathbf{v} = \mathbf{e}_z \times \boldsymbol{\nabla} \psi \ ; \quad \omega = \Delta \psi, \tag{1}$$

where the solenoidal (incompressible) velocity $\mathbf{v}$ is expressed as the orthogonal of the streamfunction gradient $\nabla\psi$. We complement the equation $\omega = \Delta\psi$ with impenetrability boundary conditions: $\psi = 0$ on $\partial\mathcal{D}$, where $\mathcal{D}$ is a simply connected domain.

The kinetic energy of the flow is conserved. It reads

$$\mathcal{E}[\omega] = \frac{1}{2} \int_{\mathcal{D}} \mathrm{d}\mathbf{r}\, \mathbf{v}^2 = \frac{1}{2} \int_{\mathcal{D}} \mathrm{d}\mathbf{r}\, (\nabla\psi)^2 = -\frac{1}{2} \int_{\mathcal{D}} \mathrm{d}\mathbf{r}\, \omega\psi. \tag{2}$$

Other conservation laws are discussed in section 2.2.3.

2.2 Microcanonical measures

2.2.1 Theoretical foundations of equilibrium statistical mechanics

Let us consider a canonical Hamiltonian system: $\{q_i\}_{1\leq i\leq N}$ denote the generalized coordinates, $\{p_i\}_{1\leq i\leq N}$ their conjugate momenta, and $H(\{q_i,p_i\})$ the Hamiltonian. The variables $\{q_i,p_i\}_{1\leq i\leq N}$ belong to a $2N$-dimensional space Ω called the phase space. Each point $(\{q_i,p_i\})$ is called a microstate. The equilibrium statistical mechanics of such a canonical Hamiltonian system is based on the Liouville theorem, which states that the non-normalized measure

$$\mu = \prod_i \mathrm{d}p_i \mathrm{d}q_i$$

is dynamically invariant. The invariance of μ is equivalent to

$$\sum_i \left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right) = 0\,, \tag{3}$$

which is a direct consequence of the Hamilton equations of motion:

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i}\,, \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i}\,. \end{cases}$$

Note that the equations of motion can also be written in a Poisson bracket form:

$$\begin{cases} \dot{q}_i = \{q_i, H\}\,, \\ \dot{p}_i = \{p_i, H\}\,. \end{cases} \tag{4}$$

Each term of the sum (3) actually vanishes independently:

$$\forall\, i, \quad \frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} = 0\,.$$

Such a relation is called a detailed Liouville theorem.

For all conserved quantities $\{I_1(q,p),\dots,I_n(q,p)\}$ of the Hamiltonian dynamics, the measures

$$\mu_c = \frac{1}{Z_c} \prod_i \mathrm{d}p_i \mathrm{d}q_i\, F(I_1,\dots,I_n) \tag{5}$$

are also invariant measures, F being any function; Z_c is a normalization constant. An important question is to know which of these measures are relevant for describing the statistics of the physical system.

In the case of an isolated system, the dynamics is Hamiltonian and there is no exchange of energy or other conserved quantities with the environment. It is therefore natural to consider a measure that takes into account all these dynamical invariants as constraints. This justifies the definition of the microcanonical measure (for a given set of invariants $\{I_1(q,p),\dots,I_n(q,p)\}$):

$$\mu_m\left(I_{1_0},\dots,I_{n_0}\right) = \frac{1}{\Omega\left(I_{1_0},\dots,I_{n_0}\right)} \prod_i \mathrm{d}p_i \mathrm{d}q_i \prod_{k=1}^{n} \delta\left(I_k(q,p) - I_{k_0}\right), \tag{6}$$

where $\Omega(I_{1_0},\ldots,I_{n_0})$ is a normalization constant — for small $\{\Delta I_k\}$, $\Omega(I_{1_0},\ldots,I_{n_0})\prod_{k=1}^{n}\Delta I_k$ is the volume of the part of phase space such that for all k, $I_{k_0}\leq I_k\leq I_{k_0}+\Delta I_k$ [2]. Then, the Boltzmann entropy of the Hamiltonian system is

$$S=k_B\log\Omega\,,$$

where k_B is the well-known Boltzmann constant. When the system under consideration is not isolated, but coupled with an external bath of conserved quantities, other measures are necessary to describe properly the system by equilibrium statistical mechanics. A classical statistical mechanics result proves that the relevant functions F in (5) are then exponentials (Boltzmann factors), and measures of type (5) are referred to as canonical measures (hence the 'c' subscript). The notion of a coupling with a single thermal bath (or to a bath of other conserved quantity than energy) in statistical mechanics assumes that if the system is coupled to a single bath, the total system (bath + system) is described by a microcanonical measure. This is an essential assumption in order to derive the canonical measure. By contrast, when systems are coupled to the environment through irreversible couplings (without detailed balance), or through several baths with different thermodynamic parameters, there is no reason anymore to expect a canonical measure to describe the statistics of the system. Fluid mechanics systems, where usually the stirring processes and the dissipation mechanisms are physical phenomena of a different nature, can never be considered as coupled to a single bath of some conserved quantities (in all works so far on fluid systems, where the distribution of energy or other conserved quantity have been studied, the distribution of conserved quantities is non Gaussian).

Hence, the relevant statistical ensemble for these models is the microcanonical one. In the following, we will work only with microcanonical measures as a base for all derivations. If equilibrium statistical mechanics is relevant for slightly non-equilibrium situations in fluid mechanics problems, it will most likely not be through canonical distributions, but through microcanonical distributions.

In statistical mechanics studies, it is sometimes argued that, in the limit of an infinite number of degrees of freedom, canonical and microcanonical measures are equivalent. Thus, as canonical measures are more easily handled, they are preferred in many works. However, while the equivalence of canonical and microcanonical ensembles is very natural and usually true in systems with short-range interactions (commonly found in condensed matter theory), it is often not actually so in systems with long-range interactions, such as the 2D Euler equations (see for instance [9, 26, 18, 10, 20, 8, 29] and references therein).

In statistical mechanics, a macrostate M is a set of microstates verifying some conditions. The conditions are usually chosen such that they describe conveniently the macroscopic behavior of the physical systems through a reduced number of variables. For instance, in a magnetic system, a macrostate M could be the ensemble of microstates with a given value of the total magnetization; in the case of a gas, a macrostate could be the ensemble of microstates corresponding to a given local density $f\left(\mathbf{x},\mathbf{p}\right)$ in the six dimensional space $(\mathbf{x},\mathbf{p})$ (μ space), where f is defined for instance through some coarse-graining. In our fluid problem, an interesting macrostate will be the local probability distribution $\rho\left(\mathbf{x},\sigma\right)d\sigma$ to observe vorticity values $\omega\left(\mathbf{x}\right)=\sigma$ at $\mathbf{x}$ with precision $\mathrm{d}\sigma$.

[2] A more natural definition of the microcanonical measure would be the uniform measure on the submanifold defined by $I_k=I_{k_0}$ for all k. This would require adding determinants in formula (6), and imply further technical difficulties. In most cases though, in the limit of large number N of degrees of freedom, these two definitions of the microcanonical measure become equivalent. Indeed, the measures have then large-deviation properties (saddle-point evaluations), where N is the large parameter, and such determinants become irrelevant. We note that in the original works of Boltzmann and Gibbs, the microcanonical measure refers to a measure where only the energy constraint is considered.

If we identify the macrostate M with the values of the constraints that define it, we can define the probability of a macrostate $P(M)\,dM$. If the microstates are distributed according to the microcanonical measure, $P(M)$ is proportional to the volume of the subset Ω_M of phase space where microstates $\{q_i,p_i\}_{1\leq i\leq N}$ realize the state M. The Boltzmann entropy of a macrostate M is then defined to be proportional to the logarithm of the phase space volume of the subset Ω_M of all microstates $\{q_i,p_i\}_{1\leq i\leq N}$ that realize the state M.

In systems with a large number of degrees of freedom, it is customary to observe that the probability of some macrostates is concentrated close to a unique macrostate. There exist also cases where the probability of macrostates concentrates close to larger set of macrostates (see for instance [39]). Such a concentration is a very important information about the macroscopic behavior of the system. The aim of statistical physics is then to identify the physically relevant macrostates, and to determine their probability and where this probability is concentrated. This is the program we will follow in the next sections, for the 2D Euler equations.

In the preceding discussion, we have explained that the microcanonical measure is a natural invariant measure with given values of the invariants. An important issue is to know if this measure describes also the statistics of the temporal averages of the Hamiltonian system. This issue, called ergodicity will be discussed in section 4.4.

2.2.2 Hamiltonian structure for the 2D Euler equations

The first step to define the microcanonical measure is to identify the equivalent of a Liouville theorem and of the dynamical invariants. The Euler equations describe a conservative dynamics. They can be derived from a least-action principle [64, 36], just like canonical Hamiltonian systems. It is thus natural to expect a Hamiltonian structure for them as well. There are however fundamental differences between infinite-dimensional systems like the 2D Euler equations and canonical Hamiltonian systems:

1. The Euler equations consist in a dynamical system of infinite dimension. The notion of volume of an infinite-dimensional space is meaningless. Hence, the microcanonical measure cannot be defined straightforwardly.
2. For such infinite-dimensional systems, we cannot in general find a canonical structure (pair of canonically conjugated variables $\{q_i,p_i\}$ describing all degrees of freedom). It exists however a Poisson structure: one can define a Poisson bracket $\{\cdot\,,\cdot\}$, like in canonical Hamiltonian systems (4), so that the dynamics writes

   $$\partial_t\omega=\{\omega,\mathcal{H}[\omega]\}\,, \tag{7}$$

   where $\mathcal{H}$ is the Hamiltonian.

For infinite-dimensional Hamiltonian systems (such as the 2D Euler equations), the Poisson bracket in (7) is often degenerate [35, 55], leading to the existence of an infinite number of conserved quantities. These conservation laws have very important dynamical consequences, as explained in the next section. A detailed description of the Hamiltonian structure of infinite-dimensional systems is beyond the scope of this paper. We refer to [35, 55] for a description of the Poisson and Hamiltonian structure for many fluid systems. In the next two sections, the dynamical invariants and the Liouville theorem are discussed in the context of the 2D Euler equations.

2.2.3 Casimir conservation laws

2D Euler equations (1) conserve an infinite number of functionals, named Casimir invariants (or Casimirs for short). They are all functionals of the form

$$\mathcal{C}_s[\omega] = \int_{\mathcal{D}} \mathrm{d}\mathbf{r}\, s(\omega), \tag{8}$$

where s is any sufficiently smooth function. As mentioned in section 2.2.1, Casimir conserved quantities are related to the degeneracy of the Poisson structure in infinite-dimensional Hamiltonian systems. They can also be understood as the invariants arising from Noether's theorem, as a consequence of the relabeling symmetry of fluid mechanics (see for instance [64]).

Let us note $A(\sigma)$ the area of $\mathcal{D}$ with vorticity values less than σ, and $\gamma(\sigma)$ the vorticity distribution:

$$\gamma(\sigma) = \frac{1}{|\mathcal{D}|}\frac{\mathrm{d}A}{\mathrm{d}\sigma} \quad \text{with } A(\sigma) = \int_{\mathcal{D}} \mathrm{d}\mathbf{r}\, \chi_{\{\omega(\mathbf{x})\le\sigma\}}\,, \tag{9}$$

where $\chi_{\mathcal{B}}$ is the characteristic function of the set $\mathcal{B} \subset \mathcal{D}$, and $|\mathcal{D}|$ is the area of $\mathcal{D}$. Since equations (1) express transport by an incompressible flow, the area $\gamma(\sigma)$ occupied by a given vorticity level σ (or equivalently $A(\sigma)$) is a dynamical invariant.

The conservation of the distribution $\gamma(\sigma)$ is equivalent to the conservation of all Casimir functionals (8). The domain-averaged vorticity $\mathcal{G}$, enstrophy $\mathcal{G}_2$, and higher moments of the vorticity $\{\mathcal{G}_k\}_{k\ge 3}$ are Casimirs of particular interest:

$$\text{for } k \ge 1, \quad \mathcal{G}_k[\omega] = \int_{\mathcal{D}} \mathrm{d}\mathbf{r}\, \omega^k \qquad (\text{with } \mathcal{G}[\omega] := \mathcal{G}_1[\omega]). \tag{10}$$

Note that if $\mathcal{D}$ is bounded, $\mathcal{G}$ is also the circulation: $\mathcal{G} = \int_{\partial\mathcal{D}} \mathbf{v}\cdot \mathrm{d}\mathbf{l}$.

In any Hamiltonian system, symmetries are associated with conservation laws, as a consequence of Noether's theorem. Then, if the domain $\mathcal{D}$ is invariant under rotations or translations, there will be conservation of angular momentum or linear momentum respectively. If the domain displays such symmetries, these conservation laws have to be taken into account in a statistical mechanics analysis.

2.2.4 Detailed Liouville theorem and microcanonical measure for the dynamics of conservative flows

In order to discuss the detailed Liouville theorem, and build the microcanonical measure, we decompose the vorticity field on the eigenmodes of the Laplacian on $\mathcal{D}$. We could decompose the field in any other orthonormal basis. The Laplacian and Fourier bases prove simpler for the following discussion, whereas finite-element bases are much more natural to justify a mean-field approximation and to obtain large-deviation results for the measures, as will be discussed in section 2.4.

We call $\{e_i\}_{i\ge 1}$ the orthonormal family of eigenfunctions of the Laplacian on $\mathcal{D}$:

$$-\Delta e_i = \lambda_i e_i\,, \quad \int_{\mathcal{D}} \mathrm{d}\mathbf{r}\; e_i e_j = \delta_{ij}\,. \tag{11}$$

The eigenvalues $\{\lambda_i\}$ are arranged in increasing order. For instance, for a doubly periodic or infinite domain, $\{e_i\}$ are just the Fourier modes. Any function g defined on $\mathcal{D}$ can be decomposed into $g(t,\mathbf{r}) = \sum_i g_i(t)e_i(\mathbf{r})$ with $g_i(t) = \int_{\mathcal{D}} \mathrm{d}\mathbf{r}\ g(t,\mathbf{r})e_i(\mathbf{r})$. Then,

$$\omega(\mathbf{r},t) = \sum_{i=1}^{+\infty} \omega_i(t)e_i(\mathbf{r}).$$

From (1),

$$\dot{\omega}_i = A_{ijk}\omega_j\omega_k\,, \tag{12}$$

where the explicit expression of A_{ijk} need not be known for the following discussion. For (12), a detailed Liouville theorem holds:

$$\forall\, i, \quad \frac{\partial \dot{\omega}_i}{\partial \omega_i} = 0 \tag{13}$$

(see [46], [42]). Note that even though we have discussed the detailed Liouville theorem in the context of mode decomposition, more general results exist [61][72] [3].

Microcanonical measure From the detailed Liouville theorem (13), we can define the microcanonical measure. First, let us define the n-moment microcanonical measure:

$$\mu_{m,n}\left(E,\Gamma_1,\ldots,\Gamma_n\right) = \frac{1}{\Omega_n\left(E,\Gamma_1,\ldots,\Gamma_n\right)} \prod_i \mathrm{d}\omega_i\ \delta\left(\mathcal{E}\left[\omega\right]-E\right) \prod_{k=1}^{n} \delta\left(\mathcal{G}_k\left[\omega\right]-\Gamma_k\right), \tag{14}$$

where $\mathcal{E}$ is the energy (2) and $\{\Gamma_k\}$ are the vorticity moments (10), the subscript m still standing for 'microcanonical'. A precise definition of $\mu_{m,n}$ requires the definition of approximate finite-dimensional measures: for any observable ϕ_M depending on M components $\{\omega_i\}_{1\leq i\leq M}$ of ω, we define

$$\langle \mu_{m,n}^N, \phi_M \rangle = \frac{1}{\Omega_{n,N}\left(E,\Gamma_1,\ldots,\Gamma_n\right)} \int \prod_{i=1}^{N} \mathrm{d}\omega_i\ \delta\left(\mathcal{E}_N\left[\omega\right]-E\right) \prod_{k=1}^{n} \delta\left(\mathcal{G}_{k,N}\left[\omega\right]-\Gamma_k\right) \phi_M\left(\omega_1,\ldots,\omega_M\right),$$

where $\mathcal{E}_N$ and $\{\mathcal{G}_{k,N}\}$ are finite-dimensional approximations of $\mathcal{E}$ (2) and $\{\mathcal{G}_k\}$ (10) respectively. Then we define $\langle \mu_{m,n}, \phi_M \rangle = \lim_{N\to\infty} \langle \mu_{m,n}^N, \phi_M \rangle$. As explained in the next paragraph, $\Omega_{n,N}$ has usually no finite limit when N goes to infinity, so the definition of the normalization factor $\Omega_n\left(E,\Gamma_1,\ldots,\Gamma_n\right)$ in the formal notation (14) implies a proper rescaling.

$\{\mu_{m,n}\}_{n\geq 1}$ is expected to be a set of invariant measures for the 2D Euler equations (this is easily verified through formal computations). The microcanonical measure corresponding to the infinite set of invariants $\{\Gamma_k\}_{k\geq 1}$ is then defined by

$$\mu_m\left(E,\{\Gamma_k\}\right) = \lim_{n\to\infty} \mu_{m,n}\left(E,\Gamma_1,\ldots,\Gamma_n\right),$$

and is denoted by

$$\mu_m\left(E,\{\Gamma_k\}\right) = \frac{1}{\Omega\left(E,\{\Gamma_k\}\right)} \prod_{i=1}^{\infty} \mathrm{d}\omega_i\ \delta\left(\mathcal{E}\left[\omega\right]-E\right) \prod_{k=1}^{\infty} \delta\left(\mathcal{G}_k\left[\omega\right]-\Gamma_k\right). \tag{15}$$

[3] A direct consequence of the detailed Liouville theorem (13) is that any truncation of the Euler equations also verifies a Liouville theorem [42]. This result is actually much more general: any approximation of the Euler equations obtained by an L_2-projection on a finite-dimensional basis verifies a Liouville theorem (see [61]). For truncations preserving the Hamiltonian structure and a finite number of Casimir invariants, see [72].

Equilibrium Boltzmann entropy The normalization factors $\Omega_n(E,\Gamma_1,\dots,\Gamma_n)$ and $\Omega(E,\{\Gamma_k\})$ define the Boltzmann entropies

$$S_n(E,\Gamma_1,\dots,\Gamma_n)=k_B\log\Omega_n(E,\Gamma_1,\dots,\Gamma_n)\quad\text{and}\quad S(E,\{\Gamma_k\})=k_B\log\Omega(E,\{\Gamma_k\})\,. \tag{16}$$

The behavior of expressions like $\int\prod_{i=1}^{N}\mathrm{d}\omega_i\,\delta(\mathcal{E}_N[\omega]-E)\prod_{k=1}^{n}\delta(\mathcal{G}_{k,N}[\omega]-\Gamma_k)$, for large N, is expected to be typically of the form $C(N)\exp(NS_n(E,\Gamma_1,\dots,\Gamma_n))$. In the definition of Ω_n, the prefactor $C(N)$ is omitted (the entropy is defined up to a constant independent of the physical variables), so S_n and S are actually 'specific entropies' (entropies per degree of freedom).

2.3 Validity of a mean-field approach to the microcanonical measures

In the previous section, we defined the microcanonical measure for the 2D Euler equations. In this section, we give a heuristic explanation of the reason why a mean-field description of the microcanonical measure is exact and give references for more precise results (large deviations for sets of measures). At the core of our discussion about mean-field approaches, lies the result that if the microstates are distributed according to the microcanonical measure, the probability distributions of vorticity at different points are independent (product measure). This is not only a result of precise mathematical works (large deviations for sets of measures), but also the deep reason for the validity of the mean-field approach. This is also the main reason for the interest of Young measures for the 2D Euler equations [52].

Because the large deviations of sets of measures are rather technical results from probability theory, for pedagogic reasons, we study the energy-enstrophy microcanonical ensemble in section 3.1, with elementary mathematical tools. We prove in Appendix D that the correlation coefficient between $\omega(\mathbf{r})$ and $\omega(\mathbf{r}')$ is zero for the energy-enstrophy microcanonical measure. It could be proved without much difficulty that, in addition, $\omega(\mathbf{r})$ and $\omega(\mathbf{r}')$ are actually independent variables. As said above, the statistical independence of vorticity values at different points is a much more general result and is essential. Let us first analyze an extremely important implication: the possibility to quantify the phase space volume (Boltzmann entropy) through the Boltzmann–Gibbs formula.

Boltzmann entropy of a macrostate and Boltzmann–Gibbs formula A classical example where degrees of freedom can be considered independent is an ensemble of particles (say, hard spheres) undergoing collisions in the dilute limit (Boltzmann–Grad limit [69]). Microscopically, particles travel with typical velocity $\bar{v}$ and collide with each other after traveling a typical distance l, called the mean free path. Let σ be the diffusion cross-section for these collisions. One has $\sigma=\pi a^2$, where parameter a is of the order of the particle radius. The mean free path is defined as $l=1/(\pi a^2 n)$, where n is the typical particle density. The Boltzmann equation applies when the ratio a/l is small (Boltzmann–Grad limit). In the limit $a/l\to 0$, any two colliding particles can be considered independent (uncorrelated) as they come from very distant regions. This is the base of Boltzmann's hypothesis of molecular chaos (Stosszahl Ansatz). It explains why the evolution of the μ-space distribution function $f(\mathbf{x},\mathbf{p},t)$ may be described by an autonomous equation, the Boltzmann equation (the μ-space is the six-dimensional space of spatial variable $\mathbf{x}$ and momentum $\mathbf{p}$). In statistical mechanics, a macrostate M is a set of microstates verifying some conditions. The conditions are usually chosen such that they describe conveniently the macroscopic behavior of the physical system through a reduced number of variables. The Boltzmann entropy of a macrostate M is defined to be proportional to the logarithm of the phase space volume of the subset Ω_M of all microstates $\{q_i,p_i\}_{1\le i\le N}$ that realize the state M. In the case of a dilute gas, the distribution $f(\mathbf{x},\mathbf{p})$ can be identified with the

macrostate: it is the set of all possible microstates $\{\mathbf{x}_i, \mathbf{p}_i\}_{1\leq i\leq N}$ such that the number of particles in the volume element $\Delta\mathbf{x}\Delta\mathbf{p}$ around $(\mathbf{x},\mathbf{p})$ is $f(\mathbf{x},\mathbf{p})\,\Delta\mathbf{x}\Delta\mathbf{p}$ (a precise mathematical definition goes through the limit $N\to\infty$, see [33]). We note that the Boltzmann entropy of the subset of phase space with fixed invariants is the equilibrium Boltzmann entropy defined in section 2.2.4, formula (16).

There is a classical argument by Boltzmann (which can be found in any good textbook on statistical mechanics) to prove that the Boltzmann entropy of the distribution f is, up to a multiplicative constant, given by the Boltzmann–Gibbs formula:

$$\mathcal{S}[f] = -\int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p}\, f\log f. \tag{17}$$

We stress that this formula for the Boltzmann entropy is not a Gibbs entropy [4]. The essential point is that formula (17) is a valid counting of the volume of the accessible part of phase space, only when particles can be considered independent. For instance, for particles with short-range interactions studied by Boltzmann, this is valid only in the Boltzmann–Grad limit.

As discussed above, for the microcanonical measure of the 2D Euler equations, vorticity field values are independent. As we will explain below, the reason is completely different from the Boltzmann case: there is now no dilute-gas (Boltzmann–Grad) limit. Nevertheless, the consequences will be the same: if we define $\rho(\mathbf{r},\sigma)$ such that $\rho(\mathbf{r},\sigma)\,\mathrm{d}\mathbf{r}\mathrm{d}\sigma$ be the probability to have values of ω between σ and $\sigma+\mathrm{d}\sigma$ in the area element $\mathrm{d}\mathbf{r}$ around $\mathbf{r}$, then the entropy

$$\mathcal{S}[\rho] = -\int_{\mathcal{D}} \mathrm{d}\mathbf{r}\int_{-\infty}^{+\infty}\mathrm{d}\sigma\,\rho\ln\rho, \tag{18}$$

actually quantifies the phase space volume. In order to give a precise meaning of this last sentence, we first define the mean-field microcanonical variational problem.

Mean-field microcanonical variational problem As ρ is a local probability, it verifies a local normalization

$$N[\rho](\mathbf{r}) \equiv \int_{-\infty}^{+\infty}\mathrm{d}\sigma\;\rho(\sigma,\mathbf{r}) = 1. \tag{19}$$

The average vorticity, for probability density ρ, is

$$\bar{\omega}(\mathbf{r}) = \int_{-\infty}^{+\infty}\mathrm{d}\sigma\,\sigma\rho(\sigma,\mathbf{r})\,. \tag{20}$$

The average vorticity (20) is related to the average streamfunction $\bar{\psi}$ so that $\bar{\omega}=\Delta\bar{\psi}$.

The conservation of all Casimir functionals (8), or equivalently of the known vorticity distribution (9), imposes a constraint on the local probability density ρ:

$$D[\rho](\sigma) \equiv \int_{\mathcal{D}}\mathrm{d}\mathbf{r}\,\rho(\sigma,\mathbf{r}) = \gamma(\sigma)\,. \tag{21}$$

[4] The Gibbs entropy $S=-k\int\rho(p_i,q_i)\log_2(\rho(p_i,q_i))\,dp_idq_i$ is an ensemble entropy, a weight on the phase space, whereas the Boltzmann–Gibbs entropy is an integral over the μ-space. In the case of dilute gases, the Boltzmann–Gibbs entropy is just the opposite of the H function of Boltzmann. We avoid this terminology here since our discussion is not related to relaxation towards equilibrium, and because the equivalent of an H theorem has never been proved for the 2D Euler equations.

Then the mean-field entropy S of the system is given by the variational problem

$$\text{(MVP)} \qquad S(E,\gamma) = \sup_{\{\rho \mid N[\rho]=1\}} \left\{ \mathcal{S}[\rho] \mid \mathcal{E}\left[\bar{\omega}\right] = E,\, D\left[\rho\right] = \gamma \right\}.$$

where $\mathcal{E}\left[\bar{\omega}\right]$ is the energy (2) of the average vorticity field $\bar{\omega}$.

An essential point has to be noted about the energy constraint in (MVP): the constraint is expressed in terms of the average vorticity field (hence the expression, 'mean-field approximation'), meaning that correlations between vorticity values at different points are negligible, and meaning also that when computing the energy, fluctuations around the average may be neglected.

An essential point is that, up to addition of constant terms (i.e., independent of the physical parameters), the mean-field entropy (MVP) is exactly the same as the Boltzmann entropy defined from the rescaled logarithm of the phase space volume, in equation (16). The definition of the entropy (16) and the variational problem (MVP) seem so different, that the fact that they express the same concept is astonishing. This type of results is indeed one of the great achievements of statistical mechanics. In section 3.1, we show that it is verified in the case of the energy-enstrophy measure, using explicit elementary computations.

Why is the mean-field entropy equal to the Boltzmann entropy? The deep reason why vorticity field values are independent for microcanonical measures, and henceforth why entropy can be expressed by (34) can be explained rather easily at a heuristic level. Correlations between variables could appear through the dynamical constraints only: energy, Casimirs, and so on. For instance, the energy of the 2D Euler system can be expressed in a form where interactions between vorticity values appear explicitly, using the Laplacian Green function $H\left(\mathbf{r},\mathbf{r}'\right)$ (defined by $\Delta H\left(\mathbf{r},\mathbf{r}'\right) = \delta(\mathbf{r},\mathbf{r}')$ with Dirichlet boundary conditions), we have:

$$(22) \qquad \mathcal{E}[\omega] = -\frac{1}{2}\int_{\mathcal{D}} \mathrm{d}\mathbf{r}\,\omega(\Delta^{-1}\omega) = -\frac{1}{2}\int_{\mathcal{D}}\int_{\mathcal{D}} \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}'\,\omega\left(\mathbf{r}\right) H(\mathbf{r},\mathbf{r}')\omega(\mathbf{r}').$$

In formula (22) above, $H\left(\mathbf{r},\mathbf{r}'\right)$ appears as the coupling between vorticity at point $\mathbf{r}$ and vorticity at point $\mathbf{r}'$. The Laplacian Green function in a two-dimensional space is logarithmic, hence non-local. Thus, $\omega(\mathbf{r})$ is coupled to the vorticity at any other point of the domain, not only close points.

For people trained in statistical mechanics, it is natural in systems where degrees of freedom are coupled to many others, to consider these degrees of freedom statistically independent at leading order, and a mean-field approach should be a valid approximation. For example, in systems with nearest-neighbor interactions, a mean-field approach becomes exact in high dimensions, when the effective number of degrees of freedom to which one degree of freedom is coupled becomes infinite. For people not trained in statistical mechanics, this can be understood simply: when the number of coupled degrees of freedom increases to infinity, the interaction felt by one degree of freedom is no more sensitive to the fluctuations of the others, but just to their average value, owing to an effect similar to what happens for the law of large numbers. Then a mean-field treatment becomes exact, which is equivalent to saying that different degrees of freedom may be considered statistically independent.

Because of the non-locality of the Green function, the vorticity field at one point is virtually coupled to an infinite number of degrees of freedom, and then a mean-field treatment is exact. This also explains why the energy appearing in the variational problem (MVP) is computed from the average vorticity field.

To formalize the preceding heuristic explanation, in order to prove that the mean-field approximation is exact and that the Boltzmann–Gibbs formula (17) is relevant, we need a rather technical discussion. We will not explain this in detail. This was justified by theoretical physicists for the point-vortex model in the seventies (assumed to be valid by Joyce and Montgomery [38] and later proved to be self-consistent in a Kramer-Moyal expansion). In the eighties, rigorous mathematical proofs were given also for the point-vortex model (see [31, 40, 17] and references therein). In the modern formulation of statistical mechanics, the entropy appears as a large-deviation rate function for an ensemble of measures, justifying (17) and the variational problem (MVP). The proof of such large-deviation results leading to the microcanonical measure for the 2D Euler equations, justifying the mean-field approach, can be found in [52] (see also [6] and references therein).

We thus conclude that a mean-field approach to the microcanonical measure of the 2D Euler equations is valid. This justifies the use of entropy (18) and of the variational problem (MVP). This step is a crucial one as it leads to a drastic simplification compared to a direct computation from the definition of the microcanonical measure (14)–(15). The first presentation of the equilibrium statistical mechanics of the 2D Euler equations in this form dates from the beginning of the '90s with the works of Robert and Sommeria, and those of Miller [59, 54, 60, 62]. Thus, we call this theory the Robert–Sommeria—Miller (RSM) theory.

2.4 Solutions to the mean-field variational problem for the microcanonical measure

The aim of this section is to describe the critical points of the mean-field variational problem (MVP), following the first papers [59, 54, 60, 62]. For this purpose, we use the Lagrange multiplier rule to take account of the constraints: the first variations of (MVP),

$$\delta\mathcal{S} - \int_{\mathcal{D}} \mathrm{d}\mathbf{r}\, A(\mathbf{r})\delta N(\mathbf{r}) - \beta\delta\mathcal{E} - \int_{-\infty}^{+\infty} \mathrm{d}\sigma\alpha(\sigma)\delta\gamma(\sigma) = 0$$

are zero for any perturbation $\delta\rho$, where $A(\mathbf{r})$, β, and $\alpha(\sigma)$ are the Lagrange multipliers associated with the conservation of $N(\mathbf{r})$, $\mathcal{E}$ and $\gamma(\sigma)$ respectively. We obtain that the probability density distribution ρ verifies the Gibbs state equation:

$$\rho\left(\sigma,\mathbf{r}\right) = \frac{e^{\beta\sigma\bar{\psi}(\mathbf{r})-\alpha(\sigma)}}{Z_\alpha\left(\beta\bar{\psi}\left(\mathbf{r}\right)\right)} \quad \text{with} \;\; Z_\alpha\left(u\right) = \int_{-\infty}^{+\infty} \mathrm{d}\sigma\, \exp\left(\sigma u - \alpha\left(\sigma\right)\right). \tag{23}$$

We see that ρ depends on $\mathbf{r}$ through the average streamfunction $\bar{\psi}$ only. From (20) and (23), we see that there is a functional relation between the equilibrium average vorticity and the streamfunction:

$$\bar{\omega} = g\left(\beta\bar{\psi}\right) \;\; \text{with} \;\; g\left(u\right) = \frac{\mathrm{d}}{\mathrm{d}u}\log Z_\alpha(u). \tag{24}$$

This last equation characterizes the statistical equilibrium. It should be solved for any value of $(\beta, \alpha(\sigma))$. Then, one has to compute the energy and vorticity distributions as functions of β and $\alpha(\sigma)$. For given energy E and distribution $\gamma(\sigma)$, among all possible values of $(\beta, \alpha(\sigma), \rho(\sigma))$ solving (23-24), the maximizer of the entropy (MVP) is selected.

3 Energy-enstrophy microcanonical measure for the 2D Euler equations

The energy-enstrophy microcanonical measure is defined as

$$\mu_{m,K}\left(E,\Gamma_2\right)=\frac{1}{\Omega\left(E,\Gamma_2\right)}\prod_{i=1}^{\infty}\mathrm{d}\omega_i\,\delta(\mathcal{E}\left[\omega\right]-E)\delta(\mathcal{G}_2\left[\omega\right]-\Gamma_2). \tag{25}$$

This is the measure where only the quadratic invariants are taken into account. There is *a priori* no physical reason to exclude the other invariants; however, the energy-enstrophy microcanonical measure can be interesting, because it is, in some cases, a good approximation of the complete microcanonical measure. Our real motivation to treat it in detail is rather pedagogical: it will be very useful to prove with this simple example, using elementary explicit computation, the equivalence between the microcanonical measure introduced in section 2.2.4 through Fourier mode decomposition, and the solution to the microcanonical mean-field variational problem of section 2.4.

We compute the entropy and the probability distribution function for the amplitude of each mode. These computations are performed, always in the microcanonical ensemble, on one hand directly from the definition of the energy-enstrophy measure (section 3.1), and on the other hand from the mean-field variational problem (section 3.2).

The energy-enstrophy measure was treated and discussed at length by many authors in the '70s, including Kraichnan (see [42], a precise discussion can be found in [49]). However, these computations were always performed in the canonical ensemble. The following discussion gives the first derivation in the microcanonical ensemble, and the first observation of ensemble inequivalence for the energy-enstrophy ensembles — microcanonical and canonical (section 3.3). The energy-enstrophy ensembles are an elementary example of the so-called partial equivalence [29], in the theory of ensemble inequivalence. We shall come back to discuss Kraichnan-type results in section 3.4.

3.1 Direct computation of the energy-enstrophy measure from its finite-dimensional approximation

Following the discussion of section 2.2.4, the energy-enstrophy microcanonical measure is defined through N-dimensional approximations:

$$\mu_{m,K}=\lim_{N\to\infty}\mu_{m,K}^{N}\quad\text{with}\quad\mu_{m,K}^{N}=\frac{1}{\Omega_{K,N}\left(E,\Gamma_2\right)}\prod_{i=1}^{N}\mathrm{d}\omega_i\,\delta(\mathcal{E}_N\left[\omega\right]-E)\delta(\mathcal{G}_{2,N}\left[\omega\right]-\Gamma_2), \tag{26}$$

where we use the same notation as in section 2.2.4, so $2\mathcal{E}_N\left[\omega\right]=\sum_{i=1}^{N}\omega_i^2/\lambda_i$ and $\Gamma_{2,N}\left[\omega\right]=\sum_{i=1}^{N}\omega_i^2$. In the following, we assume that the first mode is non-degenerate: $\lambda_1\neq\lambda_2$ (this is always true for simply connected bounded Lipschitz domains, but this is wrong for doubly periodic boundary conditions in a square domain).

The main technical difficulty is to compute

$$\Omega_{K,N}\left(E,\Gamma_2\right)=\int\prod_{i=1}^{N}\mathrm{d}\omega_i\,\delta(\mathcal{E}_N\left[\omega\right]-E)\delta(\mathcal{G}_{2,N}\left[\omega\right]-\Gamma_2), \tag{27}$$

and the entropy

$$S_K(E,\Gamma_2) = \lim_{N\to\infty} \frac{1}{N} \log\left[\Omega_{K,N}(E,\Gamma_2)\right] - C(N,\{\lambda_i\}), \tag{28}$$

where C does not depend on the physical parameters. It depends only on N and on the geometric factors $\{\lambda_i\}$, and can be discarded as the entropy is always defined up to an arbitrary constant.

The computation of $\Omega_{K,N}$ and S_K, using representation of the delta function as an integral in the complex plane, is given in Appendix B. It yields the result

$$\Omega_{K,N}(E,\Gamma_2) \underset{N\to\infty}{\sim} C_3(N,\{\lambda_i\})\, C_4(\{\lambda_i\},\Gamma_2,N)\, \frac{\exp\left[N S_K(E,\Gamma_2)\right]}{\sqrt{2E}} \left[1 + o\left(\frac{1}{N}\right)\right],$$

$$\text{with} \quad S_K(E,\Gamma_2) = \frac{1}{2}\log\left(\Gamma_2 - 2\lambda_1 E\right) + \frac{\log 2}{2}, \tag{29}$$

(see (73) and (74), page 36), where $C_3(N,\{\lambda_i\})$ does not depend on the energy or enstrophy, and C_4 has no exponentially large contribution ($\lim_{N\to\infty}(\log C_4)/N = 0$).

We now describe finite-N effects for $\mu^N_{m,K}$ (26), the finite-N approximation of the energy-enstrophy microcanonical measure $\mu_{m,K}$ (25). It is easy to see from (26), that for the N-dimensional measure $\mu^N_{m,K}$, the distribution function for the amplitude ω_n of mode e_n is given by

$$P_{N,n}(\omega_n) = \frac{\Omega_{K,N-1;\lambda_n}\left(E - \omega_n^2/2\lambda_n, \Gamma_2 - \omega_n^2\right)}{\Omega_{K,N}(E,\Gamma_2)}, \tag{30}$$

where the definition of $\Omega_{K,N-1;\lambda_n}$ is the same as that of $\Omega_{K,N}$ (27), but with integration over ω_n excluded, and with constraint $\omega_n^2 \le \max\{2\lambda_n E, \Gamma_2\}$.

The distribution function for the energy $E_n = \omega_n^2/2\lambda_n$ of mode e_n is obtained through the change of variable $P_{N,n}(E_n)\,\mathrm{d}E_n = P_{N,n}(\omega_n)\,\mathrm{d}\omega_n$. Using result (29) for both $\Omega_{K,N-1;\lambda_1}$ (then λ_1 has to be replaced with λ_2) and $\Omega_{K,N}$, we obtain

$$P_{N,1}(E_1) \underset{N\to\infty}{\sim} C\, \frac{\exp\left[N \log\left(\Gamma_2 - 2\lambda_2 E + 2(\lambda_2 - \lambda_1)E_1\right)/2\right]}{\sqrt{E_1(E-E_1)}} \quad \text{for } 0 < E_1 < E, \tag{31}$$

and $P_{N,1}(E_1) = 0$ otherwise. C is a normalization constant which does not depend on E_1. From this expression, we can see that the most probable energy is $E_1 = E$. Moreover, the distribution is exponentially peaked around $E_1 = E$, so that in the infinite-N limit (energy-enstrophy microcanonical distribution), we have

$$P_1(E_1) = \delta(E - E_1).$$

This is a striking result: for the energy-enstrophy microcanonical measure, all the energy condensates in the first mode.

The type of result (31) is called a large-deviation result. It describes accurately the distribution function for E_1, however large the deviations from the most probable value, in the limit $N \to \infty$. If large deviations in $E - E_1$ are disregarded, a good approximation for large N of the finite-N distribution is the exponential distribution:

$$P_{N,1}(E_1) = \underset{N\to\infty}{\sim} C\, \frac{\exp\left[-N \frac{\lambda_2-\lambda_1}{\Gamma_2 - 2\lambda_1 E}(E - E_1)\right]}{\sqrt{E - E_1}} \quad \text{for } 0 < E_1 < E \text{ and } N^{1/2}(E_1 - E) \ll 1. \tag{32}$$

We note that the distribution for ω_1 is exponential as well. The amplitude of the departure of E_1 from value E is thus proportional to $1/N$ and to $\left(\Gamma_2 - 2\lambda_1 E\right) / \left(\lambda_2 - \lambda_1\right)$.

The distribution of the energy E_n of mode e_n is obtained similarly as

$$P_{N,n}\left(E_n\right) \underset{N\to\infty}{\sim} C\frac{\exp\left[N\log\left(\Gamma_2 - 2\lambda_1 E - 2\left(\lambda_n - \lambda_1\right)E_n\right)\right]}{\sqrt{E_n}} \text{ for } 0 \leq E_n \leq E. \tag{33}$$

For infinite N, the energy-enstrophy microcanonical distribution is thus a delta function with zero energy:

$$P_n\left(E_n\right) = \delta\left(E_n\right).$$

Disregarding large deviations, the finite-N distribution is also well approximated by an exponential distribution (this time a Gaussian distribution for ω_n), with typical departure from 0 of order $1/N$ for the energy and a variance of order $1/\sqrt{N}$ for ω_n. It may also be checked that for large n ($\lambda_n \gg \lambda_1$), the variance of the enstrophy becomes independent of n (asymptotic equipartition of the enstrophy).

Results such as (31)–(33) are classical in statistical mechanics: typical departures from the most probable value have a Gaussian distribution with variance of order $1/\sqrt{N}$, except for variables whose most probable values are at the edge of the accessible range. In the latter case (for instance, E_n or ω_1 in the example discussed above), the distribution is exponential with typical departure of order $1/N$.

We note that for the 2D Euler equations, only the infinite-N limit is relevant, and finite-N effects have no dynamical counterpart. They may be of interest for truncated systems only.

From the preceding discussion, we see that all the energy is concentrated in the first mode, and that the excess enstrophy $\Gamma_2 - 2\lambda_1 E$ goes to smaller an smaller scales, leading to zero energy and zero enstrophy in every mode except the first one. This condensation of energy in the first mode is the main physical prediction of the microcanonical energy-enstrophy ensemble.

3.2 Energy-enstrophy microcanonical measure from a mean-field approach

Let us compute the entropy in the (microcanonical) energy-enstrophy ensemble, now starting from the mean-field variational problem, and compare the results with those of section 3.1.

The mean-field variational problem in the (microcanonical) energy-enstrophy ensemble is the equivalent of (MVP) but with only quadratic invariants taken into account:

$$S_K\left(E,\Gamma_2\right) = \sup_{\{\rho \mid N[\rho]=1\}}\left\{\frac{1}{|\mathcal{D}|}\mathcal{S}[\rho] \mid \mathcal{E}\left[\overline{\omega}\right] = E \ , \ \int \mathrm{d}\mathbf{r}\mathrm{d}\sigma\, \sigma^2\rho = \Gamma_2\right\} \tag{34}$$

Note that we seek here to maximize the specific entropy $\mathcal{S}[\rho]/|\mathcal{D}|$, because this is what leads to the actual measure of phase space volume. It is customary in the literature to ignore the $1/|\mathcal{D}|$ prefactor for convenience, as done for instance in (MVP).

In order to compute the critical points of the constrained variational problem (34), we introduce Lagrange multipliers $A(\mathbf{r}), \beta$, and α, associated with the conservation of $N(\mathbf{r}), \mathcal{E}$, and $\mathcal{G}_2$ respectively (see (19), (2) and (10), respectively, for the expression of these quantities). Critical points of (34) are such that

$$\frac{\delta\mathcal{S}}{|\mathcal{D}|} - \int_{\mathcal{D}}\mathrm{d}\mathbf{r}\, A(\mathbf{r})\delta N(\mathbf{r}) - \beta\delta\mathcal{E} - \alpha\delta\mathcal{G}_2 = 0 \quad \forall\, \delta\rho.$$

This is equivalent to

$$\rho(\sigma, \mathbf{r}) = \rho^*(\sigma, \mathbf{r}) \equiv B(\mathbf{r}) \mathrm{e}^{|\mathcal{D}|(\beta\sigma\bar{\psi}(\mathbf{r}) - \alpha\sigma^2)},$$

where the prefactor $B(\mathbf{r}) \equiv \exp(-1 - A(\mathbf{r})|\mathcal{D}|)$ is determined from the normalization constraint:

$$\rho^*(\sigma, \mathbf{r}) = \sqrt{\frac{\alpha|\mathcal{D}|}{\pi}} \mathrm{e}^{-\alpha|\mathcal{D}|\left(\sigma - \frac{\beta\bar{\psi}(\mathbf{r})}{2\alpha}\right)^2}. \tag{35}$$

The computation above, yielding a Gaussian distribution in the energy-enstrophy ensemble, is a classical result noted in many previous works (see, for instance, [23]).

Substituting expression (35) into (18), we get

$$\frac{1}{|\mathcal{D}|} \mathcal{S}[\rho^*] = -\frac{1}{2} \log \alpha \tag{36}$$

for the expression of the entropy (see calculation in Appendix B-2). We thus conclude that the maximum-entropy solution will be the one verifying the constraints with minimum value for α.

We now compute E and Γ_2 as functions of β and α. For this, we compute the average vorticity using (35) in (20):

$$\bar{\omega}(\mathbf{r}) = \int_{-\infty}^{+\infty} \mathrm{d}\sigma \sigma \rho^*(\sigma, \mathbf{r}) = \frac{\beta}{2\alpha} \bar{\psi}(\mathbf{r}).$$

We thus have $\bar{\omega}(\mathbf{r}) = \Delta\bar{\psi}(\mathbf{r}) = \beta\bar{\psi}(\mathbf{r})/(2\alpha)$, so we can deduce that vorticity and streamfunction are proportional to a Laplacian eigenmode: for some $n \geq 1$, $\bar{\omega}(\mathbf{r}) = A\, e_n(\mathbf{r})$ and $\bar{\psi}(\mathbf{r}) = -A/\lambda_n\, e_n(\mathbf{r})$, with $\beta_n = -2\alpha_n\lambda_n$. From $\mathcal{E}[\bar{\omega}] = E$, we have $E = A^2/(2\lambda_n)$ and $\bar{\psi}(\mathbf{r}) = -\sqrt{2E/\lambda_n}\, e_n(\mathbf{r})$. From $\Gamma_2 = \int_{-\infty}^{+\infty} \mathrm{d}\sigma \sigma^2 \rho^*(\sigma, \mathbf{r})$ (see Appendix B-2 for detailed computation), we get $\Gamma_2 = 2\lambda_n E + 1/(2\alpha_n)$. Thus, we can see that the minimum value α_{n^*} of α is obtained for $n^* = 1$.

Finally, since entropy (36) is maximum for $\alpha \geq 0$ minimum, the first eigenmode ($n = 1$) is the one selected.

We are left with

$$\bar{\omega}(\mathbf{r}) = \sqrt{2\lambda_1 E}\, e_1(\mathbf{r}) \quad \text{and} \quad \alpha = \frac{1}{2\left(\Gamma_2 - 2\lambda_1 E\right)},$$

so that the equilibrium entropy is

$$S_K\left(E, \Gamma_2\right) = \frac{1}{2} \log\left(\Gamma_2 - 2\lambda_1 E\right) + \frac{\log 2}{2}.$$

Comparing this result with (29), we can conclude that the entropy computed from the mean-field variational problem, in the energy-enstrophy ensemble, is the same as the one computed directly from the definition of the energy-enstrophy microcanonical measure, through Fourier mode decomposition (finding $\bar{\omega}(\mathbf{r}) = \sqrt{2\lambda_1 E}\, e_1(\mathbf{r})$ is equivalent to finding $P_1(E_1) = \delta(E - E_1)$ and $P_n(E_n) = \delta(E_n)$).

3.3 Ensemble inequivalence

From the entropy, we can compute the inverse temperature $\beta = \partial S_K/\partial E = -\lambda_1/\left(\Gamma_2 - 2\lambda_1 E\right) \leq 0$ and fugacity $\alpha = \partial S_K/\partial\Gamma_2 = 1/\left[2\left(\Gamma_2 - 2\lambda_1 E\right)\right]$. These thermodynamical coefficients are related through $\beta = -2\lambda_1\alpha$. This relation shows that some couples of thermodynamical coefficients are not obtained in the

energy-enstrophy microcanonical ensemble, in contrast to what would be expected in the thermodynamics of classical condensed matter systems. Moreover, the determinant of the Hessian of S_K, that is, $\partial^2 S_K/\partial E^2 \cdot \partial^2 S_K/\partial \Gamma_2^2 - (\partial^2 S_K/\partial E \partial \Gamma_2)^2$, is zero, showing that S_K is not strictly concave, unlike what would be expected for an entropy in the case of short-range interacting systems. Both these properties are signs of non-equivalence between the microcanonical and canonical ensembles: the two ensembles would give different predictions (see for instance [9, 29, 26]). This case of ensemble inequivalence, for the energy-enstrophy ensembles, is actually a case of partial equivalence (see [29] for a definition).

A detailed discussion of ensemble inequivalence and related phase transitions, for statistical equilibrium with linear relation between vorticity and streamfunctions, including the case of the energy-enstrophy ensemble just discussed can be found in [70].

3.4 Comments on the Kraichnan energy-enstrophy theory

The term 'condensation' was proposed by Kraichnan from the analysis of the energy-enstrophy canonical ensembles [42]. As explained in section 2.2.1, canonical measures are not relevant for fluid systems; they may be useful only when yielding results equivalent to the ones from microcanonical measures. Kraichnan noticed this and worked nonetheless with canonical ensembles, maybe because he did not know how to perform microcanonical computations, most likely because at that time the possibility of ensemble inequivalence was nearly unknown [5]. Unfortunately, as mentioned earlier, the energy-enstrophy ensembles display an instance of partial ensemble inequivalence. These remarks explain the difficulties encountered by Kraichnan when analyzing the canonical measures, and why he wrongly concluded that a statistical mechanics approach would work only for truncated systems. Working in microcanonical ensembles actually allows to build invariant measures for the real (non-truncated) 2D Euler equations. If one were interested in truncated systems, then Kraichnan's work would remain very useful.

More importantly, when looking closely at Kraichnan's works (see, for instance, [42] page 565), one sees that in the canonical ensemble, a complete condensation of the energy on the gravest mode occurs only for specific values of the thermodynamical parameters. For most values of the thermodynamical parameters, an important part of the energy remains on the other modes. Still Kraichnan argued, probably from numerical observations available at the time and from physical insight, that these cases leading to a condensation were the most interesting ones. The microcanonical treatment we propose here proves that a complete condensation occurs whatever the values of the energy and of the enstrophy, in the microcanonical ensemble. A complete condensation is actually observed in many numerical simulations. We thus conclude that the physical insight of Kraichnan and his concept of condensation describes the relevant physical mechanism, but that a treatment in the microcanonical ensemble provides a much better understanding, and overcomes the preceding contradictions.

Limitations of the energy-enstrophy approach There is *a priori* no reason to consider only the energy and enstrophy invariants, except for being able to solve the mathematics easily. From the discussion of section (2.4), we know that a mean-field approach is exact for the microcanonical measures in the case of 2D Euler equations, and thus the description of any microcanonical measure (corresponding to any set of invariants) is not difficult.

[5] The first observation of ensemble inequivalence was made in the astrophysical context [48, 34], while a thorough study [26, 9] and understanding of the importance of ensemble inequivalence for two-dimensional flows [66, 40, 31, 29, 70] are more recent.

As has been shown in previous works (see, for instance, [12]), when taking into account all invariants, the energy will no longer be limited to the first mode e_1. The energy-enstrophy measure may still be a good approximation in some cases: in the limit of small energy, for instance, most of the energy will remain in the first few modes. The notion of condensation will thus remain valid only roughly speaking, at a qualitative level.

By contrast, in some cases such as that of doubly periodic domains with aspect ratio close to but different than one (see [12]), the notion of condensation would lead to completely wrong predictions.

4 Invariant measures of the 2D Euler equations

In the previous section, we have built microcanonical measures for the 2D Euler equations and argued that they are a special set of Young measures. In this section, we consider the dynamics of measures and more specifically the dynamics of Young measures. We give a direct proof that sets of Young measures, including microcanonical ones, are invariant measures of the 2D Euler equations. For this we derive the evolution equation verified by the characteristic functional, in section 4.1. This equation describes the evolution of all the statistics of the system: it is equivalent to the Liouville equation and includes the hierarchy of equations describing the evolution of the statistics of the vorticity field.

This section is completely independent from the previous one. The proof that sets of Young measures are invariant is independent from the building of the microcanonical measure in the previous section. However, the construction of microcanonical measures from the Liouville theorem, in the previous section, shows that among the set of Young measures, microcanonical measures have a specific meaning.

4.1 Evolution of the characteristic and cumulant-generating functionals

For any random variable x, it is customary to define the characteristic function $f(l) = \left\langle e^{ilx} \right\rangle$ and the cumulant-generating function $h(l) = \log f(l)$, where the angle brackets denote average over the measure of the random variable x. In order to describe the temporal evolution of the statistics of the vorticity field $\omega(\mathbf{r},t)$, it will prove very useful to use a generalization of the characteristic and cumulant-generating functions to random fields. We consider an ensemble of initial conditions $\{\omega_0(\mathbf{r})\}$. Each of these initial conditions evolves according to the 2D Euler equations, defining an ensemble of solutions of the 2D Euler equations $\{\omega(\mathbf{r},t)\}$. We define the characteristic and cumulant-generating functionals of the ensemble $\{\omega_0(\mathbf{r})\}$, respectively, as

$$F_0[l] = \left\langle e^{i\int l(\mathbf{r})\omega_0(\mathbf{r})\mathrm{d}\mathbf{r}} \right\rangle \text{ and } H_0[l] = \log F[l],$$

where the angle brackets denote ensemble average over realizations of ω_0. We define similarly the characteristic and cumulant-generating functionals of the ensemble $\{\omega(\mathbf{r},t)\}$, respectively, as

$$F[l,t] = \left\langle e^{i\int l(\mathbf{r})\omega(\mathbf{r},t)\mathrm{d}\mathbf{r}} \right\rangle \text{ and } H[l,t] = \log F[l,t],$$

where the angle brackets still denote ensemble average over realizations of the initial conditions ω_0.

We now use that each realization $\omega(\mathbf{r},t)$ is a solution to the 2D Euler equations (1), in order to derive the evolution equation for F. A straightforward computation, reproduced in Appendix D-1, leads to

$$\frac{\partial F}{\partial t} + i \iint \mathrm{d}\mathbf{r}'\,\mathrm{d}\mathbf{r}\ \nabla\lambda(\mathbf{r}) \cdot \mathbf{G}(\mathbf{r},\mathbf{r}') \frac{\delta^2 F}{\delta l(\mathbf{r})\delta l(\mathbf{r}')} = 0\,, \tag{37}$$

where $\mathbf{G}$ is the Green function for the velocity:

$$\mathbf{v}(\mathbf{r}) = \int \mathrm{d}\mathbf{r}'\ \mathbf{G}(\mathbf{r},\mathbf{r}')\omega(\mathbf{r}'). \tag{38}$$

We note that the evolution equation for the characteristic functional (37) is a linear equation, as is the classical Liouville equation.

The equation for the cumulant-generating functional is also obtained straightforwardly (see Appendix D-2):

$$\frac{\partial H}{\partial t} + i \iint \nabla l(\mathbf{r}) \cdot \mathbf{G}(\mathbf{r},\mathbf{r}') \left(\frac{\delta^2 H}{\delta l(\mathbf{r})\delta l(\mathbf{r}')} + \frac{\delta H}{\delta l(\mathbf{r})} \frac{\delta H}{\delta l(\mathbf{r}')} \right) \mathrm{d}\mathbf{r}'\, \mathrm{d}\mathbf{r} = 0\,. \tag{39}$$

4.2 Young measures and their dynamics

4.2.1 Young measures

We recall that Young measures are uncountable product measures: the probability distribution of the vorticity field at an arbitrary number of points $\{\mathbf{r}_k\}$ is given by the product of the independent measures $\rho(\sigma,\mathbf{r}_k)$ at each point $\mathbf{r}_k$. We note that at each point, ρ is normalized

$$\int_{-\infty}^{+\infty} \mathrm{d}\sigma\ \rho\left(\sigma,\mathbf{r}\right) = 1.$$

As we see below, the fact that vorticity values at different points are independent variables has important consequences.

The set of deterministic vorticity fields are $\omega\left(\mathbf{r}\right)$ is a special class of Young measures with $\rho\left(\sigma,\mathbf{r}\right) = \delta\left(\sigma - \omega\left(\mathbf{r}\right)\right)$.

The set of microcanonical measures described in section (2.4), is a special class of Young measures. They are defined as

$$\rho_{\beta,\{\alpha\}}(\sigma,\mathbf{r}) = \frac{1}{Z(\beta\psi\left(\mathbf{r}\right))} \mathrm{e}^{\beta\sigma\psi(\mathbf{r})-\alpha(\sigma)}, \tag{40}$$

where $Z_\alpha(u) = \int_{-\infty}^{+\infty} \mathrm{d}\sigma\ \mathrm{e}^{\sigma u-\alpha(\sigma)}$.

If the local probabilities $\rho_1(\sigma,\mathbf{r})$ and $\rho_2(\sigma,\mathbf{r})$ define two Young measures, then for any $0 \le l \le 1$, $l\rho_1 + (1-l)\rho_2$ defines a Young measure.

4.2.2 Cumulant-generating functionals for Young measures

Let us evaluate the cumulant-generating functional of a Young measure. For this purpose, we consider $h\left(l,\mathbf{r}\right)$, the cumulant-generating function of the local probability $\rho(\sigma,\mathbf{r})$ at each point $\mathbf{r}$:

$$h\left(l,\mathbf{r}\right) = \log f\left(l,\mathbf{r}\right) \ \text{ with } \ f\left(l,\mathbf{r}\right) = \int_{-\infty}^{+\infty} \mathrm{d}\sigma\ \mathrm{e}^{il\sigma}\rho(\sigma,\mathbf{r}), \tag{41}$$

and the average vorticity field

$$\bar{\omega}(\mathbf{r})=\int_{-\infty}^{+\infty}\mathrm{d}\sigma\;\sigma\rho(\sigma,\mathbf{r})=\frac{\partial h}{\partial l}\left(0,\mathbf{r}\right). \tag{42}$$

Let us now consider a sufficiently regular $\lambda(\mathbf{r})$ (for instance, Riemann-integrable for the purpose of the following discussion). The characteristic functional of the entire vorticity field is easily computed using an approximation by a finite Riemann sum:

$$F[\lambda]=\left\langle \mathrm{e}^{i\int\lambda(\mathbf{r})\omega(\mathbf{r})\mathrm{d}\mathbf{r}}\right\rangle=\lim_{N\to\infty}\left\langle \mathrm{e}^{\frac{i|\mathcal{D}|}{N}\sum_{k=1}^{N}\lambda(\mathbf{r}_k)\omega(\mathbf{r}_k)}\right\rangle,$$

where $|\mathcal{D}|$ still denotes the domain area. Since variables $\{\omega(\mathbf{r}_k)\}_k$ are independent, we use the fact that the characteristic function of a set of independent variables is equal to the product of the single-variable characteristic functions:

$$F[\lambda]=\lim_{N\to\infty}\prod_{k=1}^{N}f\left(\frac{|\mathcal{D}|\lambda\left(\mathbf{r}_k\right)}{N},\mathbf{r}_k\right),$$

so that

$$H[\lambda]=\log F[\lambda]=\lim_{N\to\infty}\sum_{k=1}^{N}h\left(\frac{i|\mathcal{D}|}{N}\lambda(\mathbf{r}_k),\mathbf{r}_k\right)=\lim_{N\to\infty}\frac{i|\mathcal{D}|}{N}\sum_{k=1}^{N}\frac{\partial h}{\partial\lambda}\left(0,\mathbf{r}_k\right)\lambda(\mathbf{r}_k)\,,$$

where we have used $h\left(0,\mathbf{r}_k\right)=0$. Then, using (42), we have

$$H[\lambda]=i\int\lambda(\mathbf{r})\bar{\omega}(\mathbf{r})\mathrm{d}\mathbf{r}. \tag{43}$$

The cumulant-generating functional is linear in λ. Actually, comparing (43) to (47), we see that for sufficiently regular $\lambda\left(\mathbf{r}\right)$, the cumulant-generating functional is the same as that of a deterministic field with vorticity the average vorticity $\bar{\omega}(\mathbf{r})$. Hence, the statistics of any observable that is obtained as the domain integral of a sufficiently smooth function of $\omega(\mathbf{r})$ depends only on the average vorticity $\bar{\omega}(\mathbf{r})$. This can be seen as an example of a law of large numbers for an infinite sum of independent variables.

In the following, we call an observable S smooth additive, if S is obtained as an integral over the vorticity field**:**

$$S=\int\mathrm{d}\mathbf{r}\,\phi(\mathbf{r})\omega(\mathbf{r}). \tag{44}$$

Then, from the preceding result, we can conclude that the distribution of any smooth additive observable S is a delta function. As an illustration, we compute the characteristic functional of the velocity field at point $\mathbf{r}$, using $\lambda(\mathbf{r}')=\lambda\mathbf{G}\left(\mathbf{r},\mathbf{r}'\right)$, where $\mathbf{G}$ is the velocity Green function (38). Then,

$$\log\left\langle \mathrm{e}^{\mathrm{i}\lambda\mathbf{v}(\mathbf{r})}\right\rangle=H[\lambda\mathbf{G}\left(\mathbf{r},\mathbf{r}'\right)]=i\lambda\int\mathrm{d}'\mathbf{r}\,\mathbf{G}\left(\mathbf{r},\mathbf{r}'\right)\bar{\omega}(\mathbf{r}')\equiv i\lambda\bar{\mathbf{v}}\left(\mathbf{r}\right). \tag{45}$$

We see that the cumulant-generating function of $\mathbf{v}\left(\mathbf{r}\right)$ is linear in λ. The velocity field has no fluctuations, and hence is a delta function centered at the average value $\bar{\mathbf{v}}\left(\mathbf{r}\right)$. Once again, this is nothing but the law of large numbers.

We can easily generalize the computation of the cumulant-generating functional (43) to classes of non-regular $\lambda(\mathbf{r})$. As an example, we consider $\lambda(\mathbf{r}) = L(\mathbf{r}) + l\delta(\mathbf{r} - \mathbf{r}_0)$, where $L(\mathbf{r})$ is Riemann-integrable and l is a scalar. A direct generalization of the computations preceding (43) leads to

$$H[L(\mathbf{r}) + l\delta(\mathbf{r} - \mathbf{r}_0)] = i \int L(\mathbf{r})\bar{\omega}(\mathbf{r})\mathrm{d}\mathbf{r} + h(l, \mathbf{r}_0).$$

Using this result with $L(\mathbf{r}) = \lambda\phi$, we can describe the joint probability of any smooth additive variable S (44) and of the vorticity at point $\mathbf{r}_0$. We conclude that S and $\omega(\mathbf{r}_0)$ are independent random variables, $\omega(\mathbf{r}_0)$ having distribution $\rho(\sigma, \mathbf{r})$ and S having a delta distribution centered at the average value $\bar{S} = \int \mathrm{d}\mathbf{r}\, \phi(\mathbf{r})\bar{\omega}(\mathbf{r})$.

This result can be extended to the joint probability distribution of the vorticity at an arbitrary number of points and of any smooth additive observable. For instance, we can conclude that for Young measures, the velocity field is a random variable independent of the vorticity field, having a delta distribution centered at the average velocity $\bar{\mathbf{v}}(\mathbf{r})$, with $\Delta\bar{\psi} = \bar{\omega}$.

4.2.3 Dynamics of Young measures

We obtain the dynamics of $h(\lambda, \mathbf{r})$, the cumulant-generating function of the vorticity field at point $\mathbf{r}$ (41), either from (39) or by direct averaging of the 2D Euler equations. Using that the velocity field is independent of the vorticity field, as explained at the end of section 4.2.2, we obtain

$$\frac{\partial h}{\partial t} + \bar{\mathbf{v}} \cdot \nabla h = 0. \tag{46}$$

From this equation, it is clear that any initial measure which is Young measure remains a Young measure over time. Moreover equation (46) describe the whole dynamics, as for Young measures, any observable can be derived from h. The dynamics of Young measures is thus rather simple.

For the dynamics of Young measures, we could have worked directly with ρ. The evolution equation for ρ is just

$$\frac{\partial \rho}{\partial t} + \bar{\mathbf{v}} \cdot \nabla \rho = 0.$$

However it is more convenient to work with H (and h) as soon as perturbations to a Young measure are considered.

4.3 Classes of invariant measures

4.3.1 Deterministic dynamical equilibria

For any dynamical system, equilibria of the deterministic equations are trivial invariant measures. This is obviously also the case for the 2D Euler equations.

Let us consider a stationary solution of the 2D Euler equations $\omega_0(\mathbf{r})$, and the associated velocity field $\mathbf{v}_0(\mathbf{r})$, such that $\mathbf{v}_0(\mathbf{r}) \cdot \nabla\omega_0(\mathbf{r}) = 0$. The associated characteristic and cumulant-generating functionals are $F_0[\lambda] = \mathrm{e}^{i \int \mathrm{d}\mathbf{r}\, \lambda(\mathbf{r})\omega_0(\mathbf{r})}$ and

$$H_0[\lambda] = i \int \mathrm{d}\mathbf{r}\, \lambda(\mathbf{r})\omega_0(\mathbf{r}) \tag{47}$$

respectively. It is easily verified that F_0 and H_0 are equilibria of (37) and (39), respectively, as expected. Indeed, the second term in the l.h.s. of equation (39) is then

$$-i\int \mathrm{d}\mathbf{r}\,\nabla\lambda(\mathbf{r})\cdot\mathbf{v}_0(\mathbf{r})\omega_0(\mathbf{r}),$$

which is trivially null. This is also trivially checked from (46).

4.3.2 Invariant Young measures

From (39) and the cumulant-generating functional (43), we see that a necessary condition for a Young measure to be invariant for the 2D Euler equations is that for any sufficiently regular $\lambda(\mathbf{r})$,

$$\int \mathrm{d}\mathbf{r}\,\nabla\lambda(\mathbf{r})\cdot\bar{\mathbf{v}}(\mathbf{r})\bar{\omega}(\mathbf{r}) = 0\,.$$

Thus, a necessary condition is that the average vorticity $\bar{\omega}(\mathbf{r})$ be a dynamical equilibrium of the 2D Euler equations. From this equation, we see that h is transported by the average velocity $\bar{\mathbf{v}}$. Then a further necessary condition for a Young measure to be invariant is that h be invariant over any streamline of the velocity $\bar{\mathbf{v}}$.

This is also a sufficient condition. Indeed, in 2D Euler equations, vorticity is just transported by the velocity field. Then, because for a Young measure, the velocity has no fluctuations, if the velocity is moreover stationary and if the distribution does not depend on the streamline, then the Young measure is invariant. Then, invariant Young measures are the ones for which h is invariant over streamlines of $\bar{\mathbf{v}}$.

A smaller class of invariant Young measures of interest is the one for which ρ depends in a functional way on the streamfunction $\psi = \bar{\psi}\,:\;\rho = \rho(\sigma, \bar{\psi}\,(\mathbf{r}))$. This property has to be self-consistent:

$$\bar{\omega} = \Delta\bar{\psi} = \int d\sigma\,\sigma\rho(\sigma, \bar{\psi}\,(\mathbf{r})).$$

4.3.3 Microcanonical measures

The set of microcanonical measures (40) is a special class of Young measures. From the previous computations, their cumulant-generating functional is

$$H[\lambda] = i\int \lambda(\mathbf{r})\bar{\omega}(\mathbf{r})\mathrm{d}\mathbf{r}\quad\text{with}\;\;\bar{\omega}(\mathbf{r}) = g\left(\bar{\psi}\,(\mathbf{r})\right)\;\;\text{and}\;\;g\,(u) = \frac{\mathrm{d}}{\mathrm{d}u}\log Z\,.$$

Because of the functional relation between vorticity and streamfunction $\bar{\omega}(\mathbf{r}) = g\left(\bar{\psi}\,(\mathbf{r})\right)$, $\bar{\omega}(\mathbf{r})$ is a dynamical equilibrium of the 2D Euler equations. In conclusion, the microcanonical measures are invariant measures of the 2D Euler equations.

4.3.4 Quasi-invariant Young measures

We now consider the class of Young measures for which $\bar{\omega}(\mathbf{r})$ is a dynamical invariant of the 2D Euler equations, but for which $h(\lambda, \mathbf{r})$ is not invariant over each streamline. Then, from (46), because velocity

$\bar{\mathbf{v}}(\mathbf{r})$ is stationary, $h(\lambda, \mathbf{r})$ is just transported along each streamline. Therefore, from a microscopic point of view, such a Young measure is not invariant, but from a macroscopic point of view, it is: any smooth additive observable of the vorticity field, including the velocity field, is invariant. We call such a measure a quasi-invariant Young measure.

4.4 Ergodicity

Section 2 describes the statistical equilibria through the variational problem (MVP). The solution of this variational problem is the most probable state and also, thanks to the large-deviation property, the state around which an overwhelming majority of states do concentrate, for the microcanonical measure. Besides, the microcanonical measure is the most natural invariant measure of the 2D Euler equations with the dynamical constraints.

Having described a natural invariant measure of the equations is an important theoretical step. Another important point would be to know if this invariant measure is the only one having the right values for the dynamical invariants. The evolution of one trajectory of the dynamical system also defines a measure (through time averaging). If we knew the invariant measure were unique, then it would mean that averaging over the microcanonical measure is equivalent to averaging over time. When this uniqueness property holds, we call the dynamical system ergodic.

Generally speaking, the ergodicity of a dynamical system is a property that is usually extremely difficult to prove. Such proofs exist only for very few extremely simple systems. Ergodicity is actually thought to be wrong in general. For instance, in Hamiltonian systems with a finite number of degrees of freedom, there often exist islands in phase space in which trajectories are trapped. The common belief in the statistical mechanics community is that those parts of phase space where the motion is trapped exist, but occupy an extremely small relative volume of the phase space, for generic systems with a large number of degrees of freedom. Apart from a few systems which were proved to be integrable, this common wisdom has successfully passed empirical tests of a century of statistical mechanics studies.

There is no reason to suspect that this general picture should be different in the case of the 2D Euler equations, in general. It is thus thought that an overwhelming number of initial conditions will have a dynamics consistent with the microcanonical measure predictions. However, similarly to most other Hamiltonian systems, the 2D Euler equations are actually non-ergodic, the proofs being extremely simple, given the discussion on invariant Young measures in the previous section.

Indeed, in section 4.1, we have proved that any Young measure for which $\bar{\omega}(\mathbf{r})$ is a stationary solution of the 2D Euler equations is either an invariant or a quasi-invariant measure. The class of invariant measures corresponding to ensemble of trajectories with given values of the invariants, is then much, much larger than the class of statistical equilibrium invariant measures with the same invariants. This proves that nontrivial sets of vorticity fields are dynamically invariant. In this restricted sense, this proves that the 2D Euler equations are not ergodic.

This theoretical argument proving non-ergodicity is in accordance with previous remarks about the phenomenology of the 2D Euler or quasi-geostrophic equations. For instance, it was observed numerically that initial conditions with localized vorticity, in large domains, remain localized (see [23] and references therein; [23] actually proposes an interesting phenomenological modification of the microcanonical measure approach to cope with this localized dynamics problem). Another example of possible non-ergodicity is

the dynamics close to stable dynamical equilibria of the equations. When trajectories come close to such equilibria, they can be trapped (frozen) as was seen in some numerical simulations. A classical argument by Isichenko [37] is that for initial conditions close to parallel flows, 'displacement in certain directions is uniformly small, implying that decaying Vlasov and 2D fluid turbulence are not ergodic'. Even if the predicted algebraic laws by Isichenko are most probably wrong, the fact that displacement in directions normal to the streamlines is uniformly small is probably right, thus being another argument for non-ergodicity.

An important point to be noted, is that the Navier-Stokes equation with stochastic forces can be proved to be ergodic [16]. This ergodicity refers to invariant measures of the Navier-Stokes equations, which are non-equilibrium invariant measures with fluxes of conserved quantity. A very important point is to understand the limit of weak forces and dissipation for such invariant measures and to study their relations with the invariant measures of the 2D Euler equations. Some very interesting results can be found in [44].

4.5 Stability of invariant measures

As discussed in the previous section, the dynamical stability of invariant measures is an essential point. In the following two sections, we give a first discussion of the dynamical stability of invariant Young measures.

We first discuss, in section 4.5.1, the stability of invariant Young measures when the perturbation is such that the initial condition is still a Young measure. We conclude that the stability then depends only on the unperturbed average vorticity $\bar{\omega}_0(\mathbf{r})$ and velocity $\bar{\mathbf{v}}_0(\mathbf{r})$. More precisely, if in the framework of the 2D Euler equations small perturbations of $\bar{\omega}(\mathbf{r})$ lead to finite Lagrangian transport in the direction transverse to the streamlines, then the Young measure is stable to perturbations among Young measures. This condition of finite Lagrangian transport cross to the streamlines, is true for the 2D linearized equations for a whole class of parallel flows, including flows with stationary streamlines [11]. Moreover, we guess it is also true for the nonlinear 2D Euler equations, for large classes of parallel flows and stable vortices, even if there are still no proofs yet.

In section 4.5.2, we consider the stability of dynamical equilibria of the 2D Euler equations, subjected to perturbations whose measure is not necessarily in the the class of Young measures. For this study, we assume perturbations to be small. We prove that the linear stability of the unperturbed flow in the framework of the 2D Euler equations, implies the linear stability in a statistical sense.

4.5.1 Stability among Young measures

As explained in section 4.2.3, the dynamics of a Young measure is equivalent to the dynamics of its cumulant-generating function $h(\lambda,\mathbf{r})$ (46), where $\bar{\mathbf{v}}(\mathbf{r})$ is the velocity field corresponding to $\bar{\omega}(\mathbf{r})$. Moreover, the set of Young measures is dynamically stable. We can thus consider stability of Young measures among the set of Young measures (we take the initial condition as a Young measure that is close to the unperturbed invariant one $h_0(\lambda,\mathbf{r})$). We note that this includes the case when each realization leading to the Young measure is perturbed by the same initial perturbation $\delta\omega(\mathbf{r})$, in which case $h(\lambda,\mathbf{r}) = h_0(\lambda,\mathbf{r}) + i\lambda\delta\omega(\mathbf{r})$.

We have seen that invariant Young measures are the ones for which $\bar{\mathbf{v}}(\mathbf{r})$ is stationary and $h(\lambda,\mathbf{r})$ is constant over every streamline of $\bar{\mathbf{v}}(\mathbf{r})$. We consider such an invariant Young measure, denoting the associated cumulant-generating function by $h_0(\lambda,\mathbf{r})$ and the associated velocity by $\bar{\mathbf{v}}_0(\mathbf{r})$, and a small

perturbation to $h_0 : h = h_0 + \delta h$. We recall (see (46)), that the dynamics is

$$\frac{\partial h}{\partial t} + \bar{\mathbf{v}} \cdot \nabla h = 0, \tag{48}$$

with $\bar{\omega}(\mathbf{r}) = \frac{\partial h}{\partial \lambda}(0, \mathbf{r})$ and $\bar{\mathbf{v}}(\mathbf{r})$ the corresponding velocity field. From (48), we have

$$\frac{\partial \bar{\omega}}{\partial t} + \bar{\mathbf{v}} \cdot \nabla \bar{\omega} = 0. \tag{49}$$

We conclude that $\bar{\omega}$ satisfies the deterministic 2D Euler equations. A necessary condition for h_0 to be stable is thus that ω_0 be stable for the 2D Euler equations.

A variety of notions of stability exist for the 2D Euler equations, depending on the norms used to control the initial conditions and the evolving solutions. As far as the initial conditions are concerned, from a physical point of view, it is very natural to consider small perturbations of the initial velocity, as small-scale vorticity fluctuations are usually not controlled. We note that (48) and (49) are readily solved using Lagrangian coordinates: defining $\mathbf{R}(t)$ by

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \mathbf{v}(\mathbf{R}) \quad \text{with } \mathbf{R}(t=0)\mathbf{r} = \mathbf{r}.$$

Then, using the incompressibility of $\mathbf{v}$, we deduce that$\mathbf{R}(t)$ is invertible at each time t, and we have: $\omega(\mathbf{r}, t) = \omega(\mathbf{R}(t)^{-1}\mathbf{r}, 0)$ and $h(\lambda, \mathbf{r}, t) = h(\lambda, \mathbf{R}(t)^{-1}\mathbf{r}, 0)$.

As discussed in paragraph 4.3.4, the motion along streamlines does not matter for the invariance of the measure. A natural definition of stability is therefore imposing that the motion normal to the streamlines remain small over time. We call this Lagrangian stability; it can be defined more precisely as, say,

$$\forall\, \epsilon,\ \exists\, \alpha : \quad \frac{1}{2}\int_{\mathcal{D}} \mathrm{d}\mathbf{r}\, (\delta \mathbf{v})^2 (t=0) \le \alpha \quad \Rightarrow \quad \| \nabla_n^{\perp} \omega_0 \cdot \mathbf{R}(t)\mathbf{r} \| \le \epsilon,$$

where $\nabla_n^{\perp}\omega_0$ is the unit vector orthogonal to $\nabla \omega_0$ and where different norms $\|.\|$ define different notions of stability.

Then, clearly, from the preceding discussion, because the dynamics of Young measures is just the transport by the average velocity, and $h(\lambda, \mathbf{r}, t) = h(\lambda, \mathbf{R}(t)^{-1}\mathbf{r}, 0)$, it is natural to define the Lagrangian stability of the Young measure, just as to be equivalent to the Lagrangian stability of the of 2D Euler equations for the average vorticity field.

As can be readily seen from the results in [11], a whole class of parallel flows, including flows with non-monotonic velocity profiles, are Lagrangian-stable, as far as the linear dynamics is concerned (if the perturbed velocity field evolves according to the linearized 2D Euler equations, then the associated Lagrangian transport in the direction transverse to the streamlines is uniformly bounded over time and proportional to the initial perturbation amplitude). Even if there is no proof yet, we guess that this is also true for the (nonlinear) 2D Euler dynamics for a large class of parallel flows or stable vortices. This would prove the stability of Young measures for the nonlinear dynamics.

Then, any further discussion needs a detailed study of the relaxation (asymptotic stability) of the 2D Euler equations, which is not available yet but will be considered in future works.

4.5.2 Stability of invariant measures to small velocity perturbations

We consider now the stability of a deterministic solution to the 2D Euler equations and the effect of small perturbations. The statistics is not limited to Young measures.

We start with the 2D Euler equation which is verified for any realization:

$$\frac{\partial \Omega}{\partial t} + \mathbf{V} \cdot \nabla \Omega = 0,$$

where $\Omega = \Omega_0 + \varepsilon\omega$, $\mathbf{V} = \mathbf{V_0} + \varepsilon\mathbf{v}$, and $\mathbf{V_0} \cdot \nabla\Omega_0 = 0$ (ε is the perturbation amplitude). Developing these equations yields

$$\frac{\partial \omega}{\partial t} + L[\omega] + \varepsilon \mathbf{v} \cdot \nabla \omega = 0, \tag{50}$$

where the operator L, giving the linearized 2D Euler equations, is defined by

$$L[\omega] = \mathbf{V_0} \cdot \nabla\omega + \mathbf{v} \cdot \nabla\Omega_0 = \mathbf{V_0} \cdot \nabla\omega + \left(\int \mathrm{d}\mathbf{r}' \ \mathbf{G}(\mathbf{r},\mathbf{r}')\omega(\mathbf{r}')\right) \cdot \nabla\Omega_0. \tag{51}$$

The evolution equation for the characteristic functional $F[\lambda] = \langle \mathrm{e}^{i\int \lambda(\mathbf{r})\omega(\mathbf{r})\mathrm{d}\mathbf{r}} \rangle$ is then

$$\frac{\partial F}{\partial t} + \iint \mathrm{d}\mathbf{r}' \,\mathrm{d}\mathbf{r} \ \nabla\lambda(\mathbf{r}) \cdot \left[\mathbf{G}(\mathbf{r},\mathbf{r}') \left(\varepsilon i \frac{\delta^2 F}{\delta\lambda(\mathbf{r})\delta\lambda(\mathbf{r}')} - \Omega_0(\mathbf{r}) \frac{\delta F}{\delta\lambda(\mathbf{r}')}\right) - \mathbf{V_0}(\mathbf{r}) \frac{\delta F}{\delta\lambda(\mathbf{r})}\right] = 0. \tag{52}$$

(a detailed derivation is provided in Appendix D). Likewise, the cumulant-generating functional $H = \ln F$ satisfies

$$\frac{\partial H}{\partial t} + \iint \mathrm{d}\mathbf{r}' \,\mathrm{d}\mathbf{r} \ \nabla\lambda(\mathbf{r}) \cdot \left[\mathbf{G}(\mathbf{r},\mathbf{r}') \left(\varepsilon i \left(\frac{\delta^2 H}{\delta\lambda(\mathbf{r})\delta\lambda(\mathbf{r}')} + \frac{\delta H}{\delta\lambda(\mathbf{r})}\frac{\delta H}{\delta\lambda(\mathbf{r}')}\right) - \Omega_0(\mathbf{r}) \frac{\delta H}{\delta\lambda(\mathbf{r}')}\right) - \mathbf{V_0}(\mathbf{r}) \frac{\delta H}{\delta\lambda(\mathbf{r})}\right] = 0. \tag{53}$$

We expand the equation for the cumulant-generating functional in powers of ϵ: $H = H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \ldots$. At lowest order ($\varepsilon^0$) we have

$$\frac{\partial H_0}{\partial t} + \mathcal{L}\left[\frac{\delta H_0}{\delta\lambda(\mathbf{r})}\right] = 0, \tag{54}$$

where the linear operator $\mathcal{L}$ is defined by

$$\begin{aligned} \mathcal{L}\left[\frac{\delta H}{\delta\lambda(\mathbf{r})}\right] &= -\int \mathrm{d}\mathbf{r} \ \nabla\lambda(\mathbf{r}) \cdot \left[\mathbf{V_0}(\mathbf{r})\frac{\delta H}{\delta\lambda(\mathbf{r})} + \Omega_0(\mathbf{r}) \int \mathrm{d}\mathbf{r}' \ \mathbf{G}(\mathbf{r},\mathbf{r}')\frac{\delta H}{\delta\lambda(\mathbf{r}')}\right] \\ &= \int \mathrm{d}\mathbf{r} \ \lambda(\mathbf{r}) L\left[\frac{\delta H}{\delta\lambda(\mathbf{r})}\right] \end{aligned} \tag{55}$$

(see equation (51)). We now remark that H_0 satisfies the same equation as the one for the cumulant-generating functional in the case of linearized 2D Euler equations.

At the linear level (first order in ϵ), the stability of the measure is thus equivalent to the stability of the operator $\mathcal{L}$. We remark that a detailed knowledge of the properties of L is sufficient to describe the properties of $\mathcal{L}$; indeed,

$$\exp\left(t\mathcal{L}\right)\left[\frac{\delta H}{\delta\lambda(\mathbf{r})}\right]=\int\mathrm{d}\mathbf{r}\;\lambda(\mathbf{r})\exp\left(tL\right)\left[\frac{\delta H}{\delta\lambda(\mathbf{r})}\right].$$

As a consequence, the stability of L ($\exp\left(tL\right)$ uniformly bounded over time) is a necessary and sufficient condition for the stability of $\mathcal{L}$ ($\exp\left(t\mathcal{L}\right)$ uniformly bounded over time). In the case of stable parallel base flow $\mathbf{V_0}=U\left(y\right)\mathbf{e}_x$, a detailed study of the asymptotic behavior of the linear operator L is provided in [11]: in particular, for any perturbation, it is proved that $\left\|\exp\left(tL\right)\right\|$ is usually composed of the contribution of few modes (often no mode, actually) plus the contribution of a continuous spectrum that decays algebraically for large times. These results hold in the cases of both monotonic and non-monotonic velocity profiles U, and are probably easily generalizable to the case of stable circular vortices.

We note that the expansion of H at higher orders could be performed easily. Discussion of the convergence of such an expansion requires a detailed study of the relaxation (asymptotic stability) of the (nonlinear) 2D Euler equations, which is not available yet and that will be considered in future works.

As already stated several times, we note that a complete theory for the stability of Young measures requires the understanding of the nonlinear relaxation of the 2D Euler equations. Even at a linear level, a more complete study of the stability of Young measures would involve investigating the effect of small perturbations on any invariant Young measure, not only on deterministic solutions as done in this section. This requires a more involved expansion than the one in this section and will also be considered later on.

5 Invariant measures of the Vlasov equation

In this section, we consider the Vlasov equation. For the sake of simplicity we consider one-dimensional physical systems: however, the discussion easily extends to any dimension. Thanks to the the theoretical similarity between the 2D Euler and the Vlasov equations, noted decades ago, all the discussions about the microcanonical measures, Young measures, invariant measures, and stability of invariant measures of sections 2, 3.1, and 4.1 easily extend to the case of the Vlasov equation. In the following sections, we only describe briefly the statistical equilibrium measures and the invariant Young measures. We also give a proof of the uniqueness of statistical equilibria in the case of repulsive convex potentials, which can be useful in many future studies.

5.1 The Vlasov equation

We consider a set of particles subjected to their mutual two-body interactions with potential W. Each particle located at point x is subject to the potential $\phi_{\mathrm{discrete}}(x)=\frac{1}{N}\sum_{i=1}^{N}W(x-x_i)$, where $\{x_i\}$ are the particle positions. W is an even function. Classical physical arguments and mathematical proofs justify that, when W is regular enough, it is natural to consider the following continuum approximation to this potential:

$$\phi(x,t)=W(x-x')f(x',p,t). \tag{56}$$

The time evolution for the one-particle phase space distribution function $f(x,p,t)$ satisfies the Vlasov equation, given by

$$\frac{\partial f}{\partial t}+p\frac{\partial f}{\partial x}-\frac{\mathrm{d}\phi}{\mathrm{d}x}\frac{\partial f}{\partial p}=0\,. \tag{57}$$

If $\{x_i\}$ were N independent random variables distributed according to the distribution f, equation (56) would then follow from the law of large numbers and would be a good approximation to ϕ_{discrete} up to corrections of order $1/\sqrt{N}$. Replacing the true discrete potential by ϕ thus amounts to neglecting correlations between particles (the equivalent of the Stosszahl Ansatz) and finite-N effects. The potential ϕ_{discrete} being replaced by an average one, namely ϕ, may be seen as a mean-field approximation to the dynamics.

As can be easily verified, the Vlasov equation (57) inherits the conservation laws of the microscopic Hamiltonian dynamics, for instance, for the energy

$$H[f]=\int \mathrm{d}x\mathrm{d}p\left[f\frac{p^2}{2}+\frac{f\phi\left[f\right]}{2}\right], \tag{58}$$

and for the linear or angular momentum, when the system has the corresponding translational or rotational symmetry, respectively.

If we define $\psi(x,p)=-(p^2/2+\phi(x))$ and

$$\mathbf{v}=\left(-\frac{\partial\psi}{\partial p},\frac{\partial\psi}{\partial x}\right)\quad\text{and}\quad\nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial p}\right),$$

then, the Vlasov equation (57) can be recast into

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla f=0\,, \tag{59}$$

with $\nabla\cdot\mathbf{v}=0$ (see equations (1)). This simple remark explains the deep analogy between the 2D Euler and the Vlasov equations. Like the vorticity for 2D flows, f is transported by an incompressible flow. This explains most of the following properties.

The Casimir functionals

$$C_s[f]=\int \mathrm{d}x\mathrm{d}p\; s\left(f(x,p,t)\right) \tag{60}$$

are invariant *for any function s*.

Let $f_m=\sup\{f\}$. We denote $\bar{A}(\sigma)$ the area of $\mathcal{D}$ with f values greater than σ, and $\gamma\left(\sigma\right)$ the vorticity distribution:

$$\gamma\left(\sigma\right)=-\frac{\mathrm{d}\bar{A}}{\mathrm{d}\sigma}\quad\text{with }\;\bar{A}\left(\sigma\right)=\int_{\mathcal{D}}\mathrm{d}\mathbf{r}\,\chi_{\{\sigma\le f(x)\le f_m\}} \tag{61}$$

(see section 2.2.3 for the definition of $\chi_{\mathcal{B}}$) [6]. The area $\gamma(\sigma)$ of a given f-level σ (or equivalently $\bar{A}(\sigma)$) is a dynamical invariant. The invariance of $\gamma\left(\sigma\right)$ is equivalent to the invariance of all Casimirs $C_s[f]$.

[6] These definitions are different from those of section 2.2.3 (2D Euler equations), because with the Vlasov equation, the area of $\mathcal{D}$ is infinite.

For the dynamical equilibria, there is no time dependence for ϕ and ψ. Then from (59) we conclude that any distribution for which the distribution functions are constant over isovalues lines of ψ are dynamical equilibria. For instance, for any f_0, distribution of the type $f(x,p) = f_0(\psi(x,p))$ are dynamical equilibria (we note that this relation has to be self-consistent as the potential defining ϕ is also computed from f).

5.2 Equilibrium statistical mechanics of the Vlasov equation

The equations for the equilibrium statistical mechanics of the Vlasov equation were first written by Lynden-Bell [48] for self-gravitating systems; extensions and discussions of the analogy with the 2D Euler equations for the equilibrium statistical mechanics were first discussed in [24]. The equilibrium statistical mechanics can be considered following exactly the same steps as for the 2D Euler equations, in section 2. Then a mean field approach will be valid and the equilibrium distribution will be a Young measure characterized by $\rho(\sigma,x,p)$ the probability distribution for f to take the value σ at the point (x,p) of the phase space, with the normalization $\int_{-\infty}^{+\infty} \mathrm{d}\sigma\,\rho(\sigma,x,p) = 1$.

Let us define the average one-particle distribution function

$$\bar{f}(x,p) \equiv \int_{-\infty}^{+\infty} \mathrm{d}\sigma\,\sigma\rho(\sigma,x,p).$$

The mean-field variational problem defining the statistical equilibrium is

$$S(E,\gamma) = \sup_{\left\{\rho \mid \int_{-\infty}^{+\infty} \mathrm{d}\sigma\,\rho(\sigma,x,p)=1\right\}} \left\{ \mathcal{S}[\rho] \mid H[\bar{f}] = E,\ D[\rho](\sigma) = \int \mathrm{d}x\mathrm{d}p\,\rho(\sigma,x,p) = \gamma(\sigma) \right\} \tag{62}$$

(see (MVP)), where $\mathcal{S}[\rho] = -\int \mathrm{d}x\mathrm{d}p\,\rho\log\rho$. The equilibrium probability density distribution then reads:

$$\rho(\sigma,x,p) = \frac{\mathrm{e}^{-\beta\sigma\left(\frac{p^2}{2}+\phi\right)-\alpha(\sigma)}}{Z_\alpha\big(-\beta\big(\frac{p^2}{2}+\phi\big)\big)}, \tag{63}$$

where $Z_\alpha(u) = \int_{-\infty}^{+\infty} \mathrm{d}\sigma\ \mathrm{e}^{\sigma u - \alpha(\sigma)}$.

5.3 Invariant measures of the Vlasov equation

Let us now look at the evolution of the characteristic functional, defined by

$$F[\lambda] = \left\langle \mathrm{e}^{i\int \mathrm{d}x\mathrm{d}p\lambda(x,p)f(x,p)} \right\rangle.$$

Using a Green function formalism, in order to keep the analogy with the 2D Euler equations, we have

$$\mathbf{v}[f](x,p) = \int \mathrm{d}x'\mathrm{d}p'\mathbf{G}(x,x';p,p')f(x',p') + (0,p),$$

where

$$\mathbf{G}(x,x';p,p') = \left(0, -\frac{\mathrm{d}W}{\mathrm{d}x}(x-x')\right).$$

By analogy with (37), we get

$$\frac{\partial F}{\partial t}+\int \mathrm{d}x\mathrm{d}p\ \lambda(x,p)p\frac{\partial}{\partial x}\left(\frac{\delta F}{\delta\lambda(x,p)}\right)+$$
$$+i\int \mathrm{d}x\mathrm{d}p\mathrm{d}x'\mathrm{d}p'\ \frac{\partial}{\partial p}\left[\frac{\mathrm{d}W}{\mathrm{d}x}(x-x')\frac{\delta^2 F}{\delta\lambda(x,p)\delta\lambda(x',p')}\right]=0\,.$$

The dynamics of Young measures, characterized either by $\rho\left(\sigma,x,p\right)$, or $h\left(\lambda,x,p\right)=\log\left(\int \mathrm{d}\sigma\,\rho\left(\sigma,x,p\right)\exp\left(i\lambda\sigma\right)\right)$, is given by

$$\frac{\partial h}{\partial t}+\mathbf{v}\cdot\nabla h=0\,.$$

The set of invariant Young measures is thus the set of Young measures such that h (or ρ) is constant over any isoline of the average particle energy $\bar{\psi}$. We can define quasi-invariant Young measures similarly to the case of the 2D Euler equations, as discussed in section 4.3.4.

As in the case of the 2D Euler equations, because the set of Young measures and hence the set of invariant measures is much larger than the set of equilibrium measures, the Vlasov equation is non-ergodic.

The discussion of the ergodicity in the framework of the Vlasov equation has a long history, starting with the works of Lynden-Bell [48]. A lot of recent works have made detailed comparisons of the prediction of the equilibrium statistical mechanics with numerical simulations [3, 4, 2, 1, 71, 47], see also a detailed discussion in [21] and references therein. The qualitative results are similar to the ones for the 2D Euler equations: whereas some cases definitely show not so good a prediction due to the equilibrium statistical mechanics, because of lack of ergodicity, in most cases equilibrium statistical mechanics provides a fairly good prediction of the final self-organized state. For instance, this theory has been used to predict the final bunching parameter of a free-electron laser [5]. Moreover, the prediction skill of the equilibrium theory is expected to be better when the dimension increases. We stress that the Vlasov equation in unbounded physical space without a confining potential presents some specific difficulties, as then the microcanonical measure is not defined.

6 Perspectives

In this paper, we have studied classes of Young measures which are invariant measures of the 2D Euler equations. These classes include microcanonical and canonical equilibrium measures, but not only. Our approach was to consider the problem directly from a dynamical perspective, by looking at the evolution equations for the characteristic functional and for the cumulant-generating functional.

Our main motivation and the interest of this approach is, first, to study the stability of invariant measures and, second, to be able to generalize the results to other dynamical systems, for instance the 2D Navier-Stokes equation with stochastic forces.

In sections 4.5.1 and 4.5.2, we began the study of the stability of invariant Young measures. We have also stressed that a more complete study of their stability requires new results about the relaxation towards dynamical equilibria of the 2D Euler equations. Very interesting results have been obtained recently for the relaxation towards dynamical equilibria of the Vlasov equation [56]. We hope that similar results will

be obtained soon for the 2D Euler equations, and that these will be used in the future to complete the study of the stability of both the 2D Euler and Vlasov equations.

The class of invariant measures we describe in this paper are no-fluxes ones. For dissipative systems, like the 2D Navier-Stokes equations with linear friction and stochastic forces, or the 2D Euler equations with linear frictions and stochastic forces, invariant measures will exist which have fluxes of conserved quantities. However, for two-dimensional flows, the energy flux is thought to converge to zero in the limit of small forcing and dissipation, by contrast to what happens for 3D flows (anomalous dissipation). There is thus the possibility that the limit of small forces and dissipation will be well-behaved. Some very interesting recent mathematical results [44] seem to give positive insight in this direction. Our hope is that the invariant Young measures described in this paper could be first-order solutions in an asymptotic expansion of flux solutions for dissipative systems. This work is a first step in this direction, by giving a dynamical proof for sets of Young measures and studying their perturbed dynamics.

Acknowledgments

This work was supported through the ANR program STATFLOW (ANR-06-JCJC-0037-01) and through the ANR program STATOCEAN (ANR-09-SYSC-014).

Appendix A: Integration in the complex plane and saddle-point approximations

In this appendix, we evaluate the asymptotic behavior, for large N, of the following integral:

$$J_N(E,\alpha) = \alpha \int_{-\infty}^{+\infty} \mathrm{d}z \; \mathrm{e}^{-2iN\alpha E z} f_N(z) \quad \text{with} \quad f_N(z) = \prod_{n=1}^{N} \left(1 - i\frac{z}{\lambda_n}\right)^{-\frac{1}{2}}, \tag{64}$$

where $\{\lambda_n\}_{n\geq 1}$ is the set of eigenvalues of the Laplacian on the domain $\mathcal{D}$.

We first study the function f_N. It has square-root singularities at $z = z_n \equiv -i\lambda_n$, with $1 \leq n \leq N$. We consider a complex determination of the square roots, such that f_N has a branch cut along the half-line $z = -ix$, with $x \geq \lambda_1$ (see figure 1). We note that each of these singularities are integrable singularities.

We look for an asymptotic expansion (for large N) of f_N . It is a classical result [25] that

$$\lambda_n \underset{n\to\infty}{\sim} \frac{4\pi}{|\mathcal{D}|} n.$$

Then

$$\sum_{n=1}^{N} \frac{1}{\lambda_n} \underset{n\to\infty}{\sim} \frac{|\mathcal{D}|}{4\pi} \ln(N),$$

and $\sum_{n=1}^{\infty} 1/\lambda_n^2$ is a convergent series.

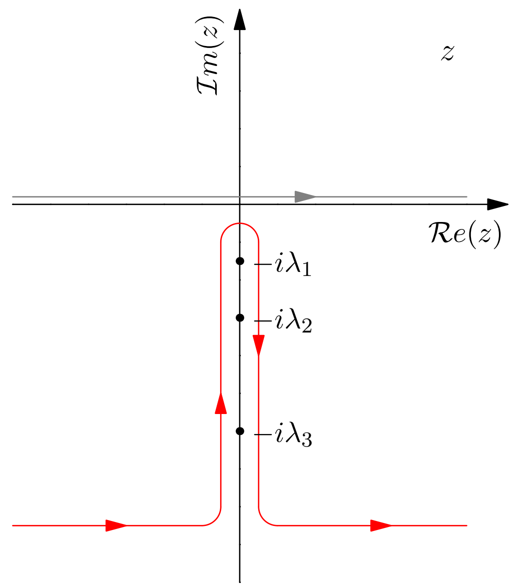

Figure 1: $f_N(z)$ (64) and integration contours used in the evaluation of integral J_N (64).

Using

$$\ln f_N(z) = -\frac{1}{2}\sum_{n=1}^{N} \ln\left(1 - i\frac{z}{\lambda_n}\right) = -\frac{1}{2}\sum_{n=1}^{N}\left[-i\frac{z}{\lambda_n} + o\left(\frac{iz}{\lambda_n^2}\right)\right],$$

we conclude

(65) $$\ln f_N(z) \underset{N\to\infty}{=} \frac{|\mathcal{D}|}{8\pi}(iz)\ln N + C(iz) + o\left(\frac{1}{N}\right) \quad \Rightarrow \quad f_N(z) \underset{N\to\infty}{\sim} N^{\frac{iz|\mathcal{D}|}{8\pi}}\tilde{C}(iz).$$

In the vicinity of $z = -i\lambda_1$, a direct extension of this result is

(66) $$f_N(-i\lambda_1 + z) \underset{N\to\infty}{\sim} \frac{N^{\frac{\lambda_1|\mathcal{D}|}{8\pi}}}{(-iz)^{\frac{1}{2}}}\tilde{C}_1(iz),$$

where $\tilde{C}$ is analytic in the vicinity of $-i\lambda_1$.

Function $f_N(z)$ is analytical except along the branch cut $z = -ix$, with $x \geq \lambda_1$. In order to compute (64), we deform the integration contour as illustrated on Figure 1: the initial and deformed contours are shown in gray and red, respectively.

It is easily checked that the contribution to J_N of the horizontal part of the contour (see Figure 1) are exponentially small, for large N. Using the change of variable $z = -i(x + \lambda_1)$, we obtain

$$J_N(E,\alpha) \underset{N\to\infty}{\sim} \alpha \mathrm{e}^{-2N\lambda_1\alpha E}\int_0^{+\infty} \mathrm{d}x\ \mathrm{e}^{-2N\alpha E x}\Delta f_N(-i(x+\lambda_1)),$$

where Δf_N is the difference between the values of f_N to the left and to the right of the branch cut. Using (65) we get

$$J_N(E,\alpha) \underset{N\to\infty}{\sim} \alpha N^{\frac{\lambda_1|\mathcal{D}|}{8\pi}}\mathrm{e}^{-2N\lambda_1\alpha E}\int_0^{+\infty} \mathrm{d}x\ \mathrm{e}^{\left(-2NE + \frac{|\mathcal{D}|}{8\pi\alpha}\ln N\right)\alpha x}\Delta\tilde{C}(-i(x+\lambda_1)).$$

For large N, this last integral is clearly dominated by values of x close to zero. Then, using (66):

$$J_N(E,\alpha) \underset{N\to\infty}{\sim} c\alpha^{1/2} N^{\frac{\lambda_1|\mathcal{D}|}{8\pi}} \mathrm{e}^{-2N\lambda_1\alpha E} \int_0^{+\infty} \mathrm{d}x \; \mathrm{e}^{\left(-2NE+\frac{|\mathcal{D}|}{8\pi\alpha}\ln N\right)x} x^{-\frac{1}{2}}$$

where $c = -2i\tilde{C}_1(0)$. Finally

$$J_N(E,\alpha) \underset{N\to\infty}{\sim} \frac{c\alpha^{1/2}}{\sqrt{2N}} N^{\frac{\lambda_1|\mathcal{D}|}{8\pi}} \frac{\mathrm{e}^{-2N\lambda_1\alpha E}}{\sqrt{E}}. \tag{67}$$

Appendix B: Energy-enstrophy microcanonical measure

B-1 Energy-enstrophy microcanonical measure from a Fourier decomposition

In this appendix, we compute the entropy for the energy-enstrophy ensemble, as well as related quantities. The entropy is defined by

$$S_K(E,\Gamma_2) = \lim_{N\to\infty}\left[\frac{1}{N}\log\Omega_{K,N}(E,\Gamma_2) - C(N,\{\lambda_n\})\right] \text{ with}$$

$$\Omega_{K,N}\left(E,\Gamma_2\right) = \int \prod_{j=1}^{N} \mathrm{d}\omega_j \, \delta(\mathcal{E}_N\left[\omega\right]-E)\delta(\Gamma_{2,N}\left[\omega\right]-\Gamma_2), \tag{68}$$

where C does not depend on the physical parameters. It depends only on N and on the geometric factors $\{\lambda_n\}$ and can be discarded as the entropy is always defined up to an arbitrary constant.

We start by relaxing the enstrophy constraint: the Dirac delta in enstrophy is thus replaced with a Boltzmann factor in the expression of $\Omega_{K,N}$. Then, we compute, for $\alpha \geq 0$,

$$I_N\left(E,\alpha\right) = \int \prod_{j=1}^{N} \mathrm{d}\omega_j \; \mathrm{e}^{-N\alpha\omega_j^2} \, \delta(\mathcal{E}_N[\omega]-E). \tag{69}$$

The relation between $I_N\left(E,\alpha\right)$ and $\Omega_{K,N}(E,\Gamma_2)$ shall be discussed at the end of this appendix. Let us use a representation as an integral in the complex plane of the remaining Dirac delta function:

$$\begin{aligned} I_N\left(E,\alpha\right) &= \int \prod_{j=1}^{N} \mathrm{d}\omega_j \; \mathrm{e}^{-N\alpha\omega_j^2} \, \delta\left(\sum_{n=1}^{N}\frac{\omega_n^2}{\lambda_n} - 2E\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}k_1 \, \mathrm{e}^{-2ik_1E} \prod_{j=1}^{N} \int \mathrm{d}\omega_j \; \mathrm{e}^{-\left(N\alpha - \frac{ik_1}{\lambda_j}\right)\omega_j^2}. \end{aligned}$$

After computing the Gaussian integrals, we get

$$I_N\left(E,\alpha\right) = \frac{\pi^{\frac{N}{2}}\alpha^{-\frac{N}{2}}}{2\pi N^{\frac{N}{2}-1}} \int_{-\infty}^{+\infty} \mathrm{d}k_1 \; \mathrm{e}^{-2iNk_1E} f_N\left(\frac{k_1}{\alpha}\right) \equiv \frac{\pi^{\frac{N}{2}}\alpha^{-\frac{N}{2}}}{2\pi N^{\frac{N}{2}-1}} J_N\left(E,\alpha\right),$$

where J_N and f_N are defined by equation (64), page 33.

Using result (67) of Appendix A, we obtain

$$I_N(E,\alpha) \underset{N\to\infty}{\sim} C_1(N,\{\lambda_n\})\, C_2(N,\{\lambda_n\},\alpha)\, \frac{\exp[-NG(E,\alpha)]}{\sqrt{2E}}, \tag{70}$$

$$\text{with} \quad G(E,\alpha) = 2\lambda_1\alpha E + \frac{\ln\alpha}{2}, \tag{71}$$

where C_1 depends only on N and $\{\lambda_n\}$ (no dependence on the physical parameters), and C_2 has no exponentially large contributions for large N ($\lim_{N\to\infty}(\ln C_2)/N = 0$).

From the definition (68) it is clear that for small $\Delta\Gamma_2$ and ΔE, $\Omega_{K,N}(E,\Gamma_2)\Delta\Gamma_2\Delta E$ is the volume of the part of phase space with energy comprised between E and $E+\Delta E$, and enstrophy between Γ_2 and $\Gamma_2+\Delta\Gamma_2$. We also note that Poincaré inequalities impose $\Gamma_{2,N}[\omega] \geq 2\lambda_1\mathcal{E}_N[\omega]$ and $\Gamma_2[\omega] \geq 2\lambda_1\mathcal{E}[\omega]$. Then from (68) and (69) we get

$$I_N(E,\alpha) = \int_{2\lambda_1 E}^{+\infty} \mathrm{d}\Gamma_2 \exp(-N\alpha\Gamma_2)\, \Omega_{K,N}(E,\Gamma_2). \tag{72}$$

It is not difficult to make for $\Omega_{K,N}(E,\Gamma_2)$ the same type of complex plane representation and saddle point approximation as the one presented for I_N in Appendix A. However, the presentation of the computation would be tedious as it involves two complex auxiliary variables (similar to k_1 above). We thus do not present these computations here, but we use that a large-deviation result holds:

$$\Omega_{K,N}(E,\Gamma_2) \underset{N\to\infty}{\sim} C_3(N,\{\lambda_n\})\, C_4(\{\lambda_n\},\Gamma_2,N)\, \frac{\exp[NS_K(E,\Gamma_2)]}{\sqrt{2E}} + o\left(\frac{1}{N}\right). \tag{73}$$

Then using this last expression in (72) and performing a saddle point approximation, we conclude that the thermodynamic potential (71) of the relaxed (canonical) ensemble is related to the entropy S_K through:

$$G(E,\alpha) = \min_{\Gamma_2 \geq 2\lambda_1 E} \{-S_K(E,\Gamma_2) + \alpha\Gamma_2\}.$$

Precisely, $G(E,\cdot)$ is the Legendre-Fenchel transform of $S_K(E,\cdot)$. It is a classical result that if G has no singularities, then S_K can be computed from the inverse formula $S_K(E,\Gamma_2) = \min_{\alpha\geq 0}\{G(E,\alpha)+\alpha\Gamma_2\}$ (see any textbook on convex analysis or [8]). Using this inversion formula we get

$$S_K(E,\Gamma_2) = \frac{1}{2}\log(\Gamma_2 - 2\lambda_1 E) + \frac{\log 2}{2}. \tag{74}$$

We note that the entropy diverges for $\Gamma_2 = 2\lambda_1 E$, the minimal accessible enstrophy Γ_2 for a given energy E. This could have been expected, as only the two microscopic states $\omega = \pm\sqrt{2E}e_1$ verify the relation $\Gamma_2 = 2\lambda_1 E$, as can be readily seen from the Poincaré inequality.

B-2 Computation of the entropy from the Boltzmann-Gibbs entropy

From (18) and (35) we the entropy of the equilibrium state ρ^* of the energy-enstrophy microcanonical measure

$$
\begin{aligned}
\frac{1}{|\mathcal{D}|}\mathcal{S}[\rho^\star] &= -\sqrt{\frac{\alpha}{\pi|\mathcal{D}|}}\int_{\mathcal{D}}\mathrm{d}\mathbf{r}\int_{-\infty}^{+\infty}\mathrm{d}\sigma\,\mathrm{e}^{-\alpha|\mathcal{D}|\left(\sigma-\frac{\beta\bar{\psi}(\mathbf{r})}{2\alpha}\right)^2}\left[\ln\left(\sqrt{\frac{\alpha|\mathcal{D}|}{\pi}}\right)-\alpha|\mathcal{D}|\left(\sigma-\frac{\beta\bar{\psi}(\mathbf{r})}{2\alpha}\right)^2\right] \\
&= -\sqrt{\frac{\alpha}{\pi|\mathcal{D}|}}\int_{\mathcal{D}}\mathrm{d}\mathbf{r}\int_{-\infty}^{+\infty}\mathrm{d}\sigma\,\mathrm{e}^{-\alpha|\mathcal{D}|\sigma^2}\left[\frac{1}{2}\ln\left(\frac{\alpha|\mathcal{D}|}{\pi}\right)-\alpha|\mathcal{D}|\sigma^2\right] \\
&= -\frac{1}{2}\left[\ln\alpha+\ln\frac{|\mathcal{D}|}{\pi}\right]+\frac{1}{2}\,.
\end{aligned}
$$

The last two terms of the rhs being generic (the entropy being defined up to a constant), we retain

$$
\frac{1}{|\mathcal{D}|}\mathcal{S}[\rho^\star] = -\frac{1}{2}\log\alpha. \tag{75}
$$

Appendix C: Correlation of the vorticity field

Here, we sketch the evaluation of the order of magnitude of the two-point correlation function, for the vorticity field obtained from the energy-enstrophy microcanonical measure. From (30), we see that, for the measure $\mu^N_{m,K}$ (26), the variance $\langle\langle\omega_n^2\rangle\rangle_N = \left\langle\omega_n^2 - \langle\omega_n\rangle_N^2\right\rangle_N$ of ω_n is of order $1/N$. Following the reasoning that lead to equations (30), the joint probability distribution $P_{N,n,m}(\omega_n,\omega_m)$ (for amplitude of modes e_n and e_m) can be derived from (27):

$$
P_{N,n,m}(\omega_n,\omega_m) \underset{N\to\infty}{\sim} C\exp\left[N\log\left(\Gamma_2-2\lambda_1 E-\frac{(\lambda_n-\lambda_1)}{\lambda_1}\omega_n^2-\frac{(\lambda_m-\lambda_1)}{\lambda_1}\omega_m^2\right)\right]. \tag{76}
$$

From this expression, for $n\neq m$, the correlation $\langle\langle\omega_n\omega_m\rangle\rangle_N = \left\langle\omega_n\omega_m - \langle\omega_n\rangle_N\langle\omega_m\rangle_N\right\rangle_N$ can be shown to be of order $1/N^2$. Now, $\omega(\mathbf{r}) = \sum_n \omega_n e_n(\mathbf{r})$, so $\langle\langle\omega(\mathbf{r})\omega(\mathbf{r}')\rangle\rangle_N$ is of order $1/N$. Therefore, for the energy-enstrophy microcanonical measure $\mu_{m,K}$ (N going to infinity),

$$
\langle\langle\omega(\mathbf{r})\omega(\mathbf{r}')\rangle\rangle = 0\,. \tag{77}
$$

We just considered the second moment of the vorticity field. However, such a result is much more general: vorticity values at points $\mathbf{r}$ and $\mathbf{r}'$ are actually statistically independent for the microcanonical measure, as could be easily shown by extending the results (76) and (77).

Appendix D: Evolution of the characteristic and cumulant-generating functionals

In this appendix, we derive the evolution equations for the characteristic and cumulant-generating functionals.

D-1 Characteristic functional

For the 2D Euler equations

In order to compute the evolution equation for the characteristic functional $F[\lambda] = \langle \mathrm{e}^{i\int \lambda(\mathbf{r})\omega(\mathbf{r})\mathrm{d}\mathbf{r}} \rangle$, we use the intermediate quantity

$$A := \mathrm{e}^{i\int \lambda(\mathbf{r})\omega(\mathbf{r})\mathrm{d}\mathbf{r}}.$$

Using the Euler equation (1) and an integration by parts, we obtain

$$\frac{\partial A}{\partial t} = iA - \int \mathrm{d}\mathbf{r}\ \omega(\mathbf{r})\mathbf{v}(\mathbf{r}) \cdot \nabla\lambda(\mathbf{r}).$$

We then use

$$\langle \mathbf{v}(\mathbf{r})\omega(\mathbf{r})A \rangle = \int \mathrm{d}\mathbf{r}'\ \mathbf{G}\left(\mathbf{r},\mathbf{r}'\right) \langle \omega\left(\mathbf{r}'\right)\omega\left(\mathbf{r}\right) A\rangle, \tag{78a}$$

$$\text{but} \quad \langle \omega(\mathbf{r}')\omega(\mathbf{r})A \rangle = -\frac{\delta^2 F}{\delta\lambda(\mathbf{r})\delta\lambda(\mathbf{r}')}\,, \tag{78b}$$

so that

$$\frac{\partial F}{\partial t} + i \iint \mathrm{d}\mathbf{r}'\,\mathrm{d}\mathbf{r}\ \nabla\lambda(\mathbf{r}) \cdot \mathbf{G}(\mathbf{r},\mathbf{r}') \frac{\delta^2 F}{\delta\lambda(\mathbf{r})\delta\lambda(\mathbf{r}')} = 0. \tag{79}$$

For the perturbation of an equilibrium of the 2D Euler equations

We now apply the same tools to the case of the 2D Euler equations near a dynamical equilibrium. Consider Ω a solution to the 2D Euler equations, with $\Omega = \Omega_0 + \varepsilon\omega$, $\mathbf{V} = \mathbf{V_0} + \varepsilon\mathbf{v}$ and $\mathbf{V_0} \cdot \nabla\Omega_0 = 0$ (see section 4.5.2).

We want to determine the evolution equation for the characteristic functional $F[\lambda] = \langle \mathrm{e}^{i\int \lambda(\mathbf{r})\omega(\mathbf{r})\mathrm{d}\mathbf{r}} \rangle$, just as we did in the case of the 2D Euler equations. Defining $A := \mathrm{e}^{i\int \lambda(\mathbf{r})\omega(\mathbf{r})\mathrm{d}\mathbf{r}}$ and using (50), page 28, we get

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= -i \int \lambda \left(\mathbf{v} \cdot \nabla\Omega_0 + \mathbf{V_0} \cdot \nabla\omega + \varepsilon\mathbf{v} \cdot \nabla\omega \right) A \\ &= i \int \nabla\lambda \cdot \left(\mathbf{v}\Omega_0 + \mathbf{V_0}\omega + \varepsilon\mathbf{v}\omega \right) A\,. \end{aligned}$$

Since

$$\langle \mathbf{v}(\mathbf{r})A \rangle = \int \mathrm{d}\mathbf{r}'\ \mathbf{G}\left(\mathbf{r},\mathbf{r}'\right) \langle \omega\left(\mathbf{r}'\right) A\rangle = -i \int \mathrm{d}\mathbf{r}'\ \mathbf{G}\left(\mathbf{r},\mathbf{r}'\right) \frac{\delta F'}{\delta\lambda(\mathbf{r})}, \tag{80}$$

we conclude

$$\frac{\partial F}{\partial t} + \iint \nabla\lambda(\mathbf{r}) \cdot \left[\mathbf{G}(\mathbf{r},\mathbf{r}') \left(\varepsilon i \frac{\delta^2 F}{\delta\lambda(\mathbf{r})\delta\lambda(\mathbf{r}')} - \Omega_0(\mathbf{r}) \frac{\delta F}{\delta\lambda(\mathbf{r}')} \right) - \mathbf{V_0}(\mathbf{r}) \frac{\delta F}{\delta\lambda(\mathbf{r})} \right]\ \mathrm{d}\mathbf{r}'\,\mathrm{d}\mathbf{r} = 0. \tag{81}$$

D-2 Cumulant-generating functional

By definition, the cumulant-generating functional is $H = \ln F$. Therefore

$$\frac{\partial H}{\partial t} = \frac{1}{F}\frac{\partial F}{\partial t}, \quad \frac{\delta H}{\delta\lambda(\mathbf{r})} = \frac{1}{F}\frac{\delta F}{\delta\lambda(\mathbf{r})}, \text{ and } \frac{\delta^2 H}{\delta\lambda(\mathbf{r})\delta\lambda(\mathbf{r}')} = \frac{1}{F}\frac{\delta^2 F}{\delta\lambda(\mathbf{r})\delta\lambda(\mathbf{r}')} - \frac{1}{F^2}\frac{\delta F}{\delta\lambda(\mathbf{r})}\frac{\delta F}{\delta\lambda(\mathbf{r}')}.$$

Hence, using (38),

$$\frac{\partial H}{\partial t} + i\iint \nabla\lambda(\mathbf{r})\cdot\mathbf{G}(\mathbf{r},\mathbf{r}')\left(\frac{\delta^2 H}{\delta\lambda(\mathbf{r})\delta\lambda(\mathbf{r}')} + \frac{\delta H}{\delta\lambda(\mathbf{r})}\frac{\delta H}{\delta\lambda(\mathbf{r}')}\right)\mathrm{d}\mathbf{r}'\,\mathrm{d}\mathbf{r} = 0.$$

For a small perturbation of a dynamical equilibrium, using (38), we get

$$\frac{\partial H}{\partial t} + \iint \nabla\lambda(\mathbf{r})\cdot\left[\mathbf{G}(\mathbf{r},\mathbf{r}')\left(\varepsilon i\left(\frac{\delta^2 H}{\delta\lambda(\mathbf{r})\delta\lambda(\mathbf{r}')} + \frac{\delta H}{\delta\lambda(\mathbf{r})}\frac{\delta H}{\delta\lambda(\mathbf{r}')}\right) - \Omega_0(\mathbf{r})\frac{\delta H}{\delta\lambda(\mathbf{r}')}\right) - \mathbf{V_0}(\mathbf{r})\frac{\delta H}{\delta\lambda(\mathbf{r})}\right]\mathrm{d}\mathbf{r}'\,\mathrm{d}\mathbf{r} = 0.$$

Appendix E: Uniqueness of the Vlasov statistical equilibria for repulsive convex potentials

In this appendix, we prove that, for a repulsive convex potential W, the microcanonical variational problem for the Vlasov equation has a unique solution. This has the following consequences: for a repulsive potential, no phase transition exists and no ensemble inequivalence exists. The argument is extremely simple; it is based on the concavity of $\mathcal{G}[\rho] = \mathcal{S}[\rho] - \beta H - \int \mathrm{d}\sigma\mathrm{d}x\mathrm{d}p\,\alpha(\sigma)\rho(\sigma,x,p)$. The concavity of $\mathcal{G}$ implies the uniqueness of the critical point of $\mathcal{G}$ (the uniqueness of the equilibrium in the grand canonical ensemble where β and α are the control parameters) and thus the absence of phase transitions in the grand-canonical ensemble. Then a classical result of convex analysis [63] (see also a simple discussion in [8]) implies that there is a one-to-one relation between the constraints $(E,\gamma(\sigma))$ and the Lagrange multipliers $(\beta,\alpha(\sigma))$. Then for any energy E and distribution $\alpha(\sigma)$, the microcanonical variational problem has a unique solution and there is no phase transition either in the microcanonical ensemble.

As $\mathcal{S}$ is strictly concave and $\int \mathrm{d}\sigma\mathrm{d}x\mathrm{d}p\,\gamma(\sigma)\rho(\sigma,x,p)$ is linear, in order to prove the strict concavity of $\mathcal{G}$ it is sufficient to prove that $-\beta H$ is concave.

It is well-known that for systems with kinetic energy $E_c = p^2/2$, the inverse temperature β is positive. That this is necessary can be seen directly from the normalizability of the equation for the critical state (63). Systems with possible negative temperature states are the ones with bounded phase space (see for instance the case of the point vortex model [57]). Then, in order to prove the concavity of $-\beta H$ it is sufficient to prove the convexity of H (58), or equivalently to prove the convexity of the potential

$$V[f] = \frac{1}{2}\int \mathrm{d}x\mathrm{d}p\, f\phi[f] = \frac{1}{2}\int \mathrm{d}x\mathrm{d}x'\, m(x)m(x')W(x-x'),$$

with $m(x) = \int \mathrm{d}p\, f(x,p)$. We also remark that if the system were confined by some external potential, because this would appear as a linear term in the functional, the convexity of H would not be affected.

The second-order variations of the potential read

$$\delta^2 V = \frac{1}{2}\int \mathrm{d}x\mathrm{d}x'\,\delta m(x)\delta m(x')W(x-x').$$

Resorting to Fourier transforms, $\delta m(x) = \int \mathrm{d}k\,\delta m_k \mathrm{e}^{ikx}$, we see that convexity of V is equivalent to positivity of W_k, the Fourier transform of W:

$$W_k = \int_0^{+\infty} \mathrm{d}x\,W(x)\cos(kx);$$

recalling that W is even.

We have $W_k = \sum_{n=0}^{\infty} W_{k,n}$, where

$$\begin{aligned}
W_{k,n} &= \int_{\frac{2n\pi}{k}}^{\frac{2(n+1)\pi}{k}} \mathrm{d}x\,W(x)\cos(kx)\\
&= \frac{1}{k}\int_0^{\frac{\pi}{2}} \mathrm{d}x\,\cos(x)\left[W\left(\frac{x+2n\pi}{k}\right) - W\left(\frac{\pi-x+2n\pi}{k}\right) - W\left(\frac{\pi+x+2n\pi}{k}\right) + W\left(\frac{2\pi-x+2n\pi}{k}\right)\right]\\
&= \frac{1}{k}\int_0^{\frac{\pi}{2}} \mathrm{d}x\,\cos(x)\int_{\frac{x+2n\pi}{k}}^{\frac{\pi-x+2n\pi}{k}} dy\,\left[W'(y+\pi) - W'(y)\right].
\end{aligned}$$

Now, using that W is convex, W' is increasing and thus $W_{k,n} \geq 0$. Hence $\forall\,k\;\; W_k \geq 0$ and the potential energy functional is convex. Therefore, H is convex, and the uniqueness of the statistical equilibria is proved.

The same type of arguments could be derived in dimensions d larger than 1. We treat, for example, the case $d = 3$. Assuming the interaction potential to be isotropic, $W = W(r)$, we have

$$W(\mathbf{k}) \equiv \frac{1}{(2\pi)^3}\int \mathrm{d}\mathbf{r}\,W(\mathbf{r})\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} = \frac{1}{(2\pi)^2 k}\int_0^{+\infty} \mathrm{d}r\,rW(r)\sin(kr).$$

Clearly, we can use the same reasoning as before, applying it to $rW(r)$ rather than to $W(x)$. We thus conclude that, in dimension 3, if W is a repulsive isotropic potential and if $rW(r)$ is convex, then $\forall\;\mathbf{k}\;\;W(\mathbf{k}) \geq 0$, the potential energy functional is convex and hence, for any value of the constraints, the statistical equilibria are unique and no phase transition exists.

Let us discuss the special case of algebraic potentials $W(r) = C/r^{\alpha}$ in dimension 3. We are interested in systems with long-range interactions (non-integrable potentials); then we suppose $\alpha \leq 2$. The condition $rW(r)$ convex is then $\alpha \geq 1$ (for $\alpha < 1$, the Fourier transform of the potential would not be defined). Then the preceding argument applies to values of α with $1 \leq \alpha \leq 2$, including for instance Coulomb potentials.

References

[1] A. Antoniazzi, Y. Elskens, D. Fanelli, and S. Ruffo, *Statistical mechanics and Vlasov equation allow for a simplified Hamiltonian description of Single-Pass Free Electron Laser saturated dynamics*, Eur. Phys. J. B **50** (2006), 603–611.

[2] A. Antoniazzi, D. Fanelli, J. Barré, P.-H. Chavanis, T. Dauxois, and S. Ruffo, *Maximum entropy principle explains quasistationary states in systems with long-range interactions: The example of the Hamiltonian mean-field model*, Phys. Rev. E**75** (2007), no. 1, 011112–+.

[3] I. Arad and P. H. Johansson, *A numerical comparison of theories of violent relaxation*, Mon. Not. R. Astron. Soc. **362** (2005), 252–262.

[4] I. Arad and D. Lynden-Bell, *Inconsistency in theories of violent relaxation*, Mon. Not. R. Astron. Soc. **361** (2005), 385–395.

[5] J. Barré, T. Dauxois, G. de Ninno, D. Fanelli, and S. Ruffo, *Statistical theory of high-gain free-electron laser saturation*, Phys. Rev. E **69** (2004), no. 4, 045501.

[6] C. Boucher, R. S. Ellis, and B. Turkington, *Spatializing Random Measures: Doubly Indexed Processes and the Large Deviation Principle* , Annals Prob. **27** (1999), 297–324.

[7] ———, *Derivation of maximum entropy principles in two-dimensional turbulence via large deviations*, J. Stat. Phys. **98** (2000), no. 5-6, 1235.

[8] F. Bouchet, *Simpler variational problems for statistical equilibria of the 2d euler equation and other systems with long range interactions*, Physica D Nonlinear Phenomena **237** (2008), 1976–1981.

[9] F. Bouchet and J. Barré, *Classification of Phase Transitions and Ensemble Inequivalence, in Systems with Long Range Interactions*, Journal of Statistical Physics **118** (2005), 1073–1105.

[10] F. Bouchet, S. Gupta, and D. Mukamel, *Thermodynamics and dynamics of systems with long-range interactions*, Physica A (2010), 4389–4405.

[11] F. Bouchet and H. Morita, *Large time behavior and asymptotic stability of the 2D Euler and linearized Euler equations*, Physica D Nonlinear Phenomena **239** (2010), 948–966.

[12] F. Bouchet and E. Simonnet, *Random Changes of Flow Topology in Two-Dimensional and Geophysical Turbulence*, Physical Review Letters **102** (2009), no. 9, 094504–+.

[13] F. Bouchet and J. Sommeria, *Emergence of intense jets and Jupiter's Great Red Spot as maximum-entropy structures*, Journal of Fluid Mechanics **464** (2002), 165–207.

[14] J. Bourgain, *Periodic nonlinear Schrödinger equation and invariant measures*, Comm. Math. Phys. **166** (1994), no. 1, 1–26.

[15] ———, *Invariant Measures for NLS in Infinite Volume*, Comm. Math. Phys. **210** (2000), no. 3, 605–620.

[16] J. Bricmont, A. Kupiainen, and R. Lefevere, *Ergodicity of the 2D Navier-Stokes Equations with Random Forcing*, Com.. Math. Phys. **224** (2001), 65–81.

[17] E. Caglioti, P. L. Lions, C. Marchioro, and M. Pulvirenti, *A special class of stationary flows for two-dimensional euler equations: A statistical mechanics description. Part II*, Commun. Math. Phys. **174** (1995), 229–260.

[18] A. Campa, T. Dauxois, and S. Ruffo, *Statistical mechanics and dynamics of solvable models with long-range interactions*, Phys. Rep. **480** (2009), 57–159.

[19] P.-H. Chavanis, *Statistical mechanis of two-dimensional vortices and stellar systems*, Dynamics and Thermodynamics of Systems With Long Range Interactions (T. Dauxois, S. Ruffo, E. Arimondo, and M. Wilkens, eds.), Lecture Notes in Physics, vol. 602, Springer-Verlag, 2002, pp. 208–289.

[20] ———, *Phase Transitions in Self-Gravitating Systems*, International Journal of Modern Physics B **20** (2006), 3113–3198.

[21] ———, *Dynamics and thermodynamics of systems with long-range interactions: interpretation of the different functionals*, Dynamics and Thermodynamics of Systems with Long Range Interactions: Theory and Experiments (A. Campa, A. Giansanti, G. Morigi, & F. S. Labini, ed.), American Institute of Physics Conference Series, vol. 970, American Institute of Physics, January 2008, pp. 39–90.

[22] P.-H. Chavanis and M. Lemou, *Relaxation of the distribution function tails for systems described by Fokker-Planck equations*, Phys. Rev. E **72** (2005), no. 6, 061106.

[23] P. H. Chavanis and J. Sommeria, *Classification of robust isolated vortices in two-dimensional hydrodynamics*, Journal of Fluid Mechanics **356** (1998), 259–296.

[24] P.-H. Chavanis, J. Sommeria, and R. Robert, *Statistical Mechanics of Two-dimensional Vortices and Collisionless Stellar Systems*, Astro. Phys. Jour. **471** (1996), 385.

[25] R. Courant and Hilbert D., *Methods of Mathematical Physics*, vol. 1, Wiley-Interscience, New York, 1953.

[26] T. Dauxois, S. Ruffo, E. Arimondo, and M. Wilkens, *Dynamics and thermodynamics of systems with long-range interactions*, Lecture Notes in Physics, Berlin Springer Verlag, vol. 602, 2002.

[27] D. H. E. Dubin, *Collisional diffusion in a two-dimensional point vortex gas or a two-dimensional plasma*, Phys. Plasmas **10** (2003), 1338–1350.

[28] D. H. E. Dubin and T. M. O'Neil, *Two-dimensional guiding-center transport of a pure electron plasma*, Phys. Rev. Lett. **60** (1988), no. 13, 1286–1289.

[29] R. S. Ellis, K. Haven, and B. Turkington, *Large Deviation Principles and Complete Equivalence and Nonequivalence Results for Pure and Mixed Ensembles*, J. Stat. Phys. **101** (2000), 999.

[30] G. Eyink, U. Frisch, R. Moreau, and A. Sobolevski, *Euler: 250 years on*, Physica D, vol. 237, 2008.

[31] G. L. Eyink and H. Spohn, *Negative-temperature states and large-scale, long-lived vortices in two-dimensional turbulence*, Journal of Statistical Physics **70** (1993), 833–886.

[32] G. L. Eyink and K. R. Sreenivasan, *Onsager and the theory of hydrodynamic turbulence*, Rev. Mod. Phys. **78** (2006), 87–135.

[33] S. Goldstein and J. L. Lebowitz, *On the (Boltzmann) entropy of non-equilibrium systems*, Physica D **193** (2004), 53.

[34] W. Hertel, P. Thirring, *Soluble model for a system with negative specific heat*, Annals Phys. **63** (1971), 520–533.

[35] D. D. Holm, J. E. Marsden, T. Ratiu, and A. Weinstein, *Nonlinear stability of fluid and plasma equilibria*, Phys. Rep. **123** (1985), 1–2.

[36] D. D. Holm, J. E. Marsden, and T. S. Ratiu, *The Euler-Poincare Equations and Semidirect Products with Applications to Continuum Theories*, eprint arXiv:chao-dyn/9801015, January 1998, pp. 1015–+.

[37] M. B. Isichenko, *Nonlinear Landau Damping in Collisionless Plasma and Inviscid Fluid*, Physical Review Letters **78** (1997), 2369–2372.

[38] G. Joyce and D. Montgomery, *Negative temperature states for the two-dimensional guiding-centre plasma*, Journal of Plasma Physics **10** (1973), 107–+.

[39] M. K. H. Kiessling, *Statistical equilibrium dynamics*, Dynamics and Thermodynamics of Systems with Long Range Interactions: Theory and Experiments (A. Campa, A. Giansanti, G. Morigi, & F. S. Labini, ed.), American Institute of Physics Conference Series, vol. 970, American Institute of Physics, January 2008, pp. 91–108.

[40] M. K. H. Kiessling and J. L. Lebowitz, *The Micro-Canonical Point Vortex Ensemble: Beyond Equivalence*, Lett. Math. Phys. **42** (1997), no. 1, 43–56.

[41] R. H. Kraichnan, *Statistical dynamics of two-dimensional flow*, Journal of Fluid Mechanics **67** (1975), 155–175.

[42] R. H. Kraichnan and D. Montgomery, *Two-dimensional turbulence*, Reports on Progress in Physics **43** (1980), 547–619.

[43] S. Kuksin and A. Shirikyan, *Stochastic Dissipative PDE's and Gibbs Measures*, Communications in Mathematical Physics **213** (2000), 291–330.

[44] S. B. Kuksin, *The Eulerian limit for 2D statistical hydrodynamics*, J. Stat. Phys. **115** (2004), 469–492.

[45] J. L. Lebowitz, H. A. Rose, and E. R. Speer, *Statistical mechanics of the nonlinear Schrödinger equation*, Journal of Statistical Physics **50** (1988), 657–687.

[46] T. D. Lee, *On some statistical properties of hydrodynamical and magnetohydrodynamical fields*, Q. Appl. Math. **10** (1952), 69–74.

[47] Y. Levin, R. Pakter, and F. B. Rizzato, *Collisionless relaxation in gravitational systems: From violent relaxation to gravothermal collapse*, Phys. Rev. E **78** (2008), no. 2, 021130–+.

[48] D. Lynden-Bell and R. Wood, *The gravo-thermal catastrophe in isothermal spheres and the onset of red-giant structure for stellar systems*, Mon. Not. R. Astron. Soc. **138** (1968), 495.

[49] A. J. Majda and X. Wang, *Nonlinear Dynamics and Statistical Theories for Basic Geophysical Flows*, Cambridge University Press, 2006.

[50] J. C. Mattingly and Y. G. Sinai, *An Elementary Proof of the Existence and Uniqueness Theorem for the Navier-Stokes Equations*, ArXiv Mathematics e-prints (1999).

[51] H. P. McKean, *Statistical mechanics of nonlinear wave equations (4): Cubic Schrödinger*, Communications in Mathematical Physics **168** (1995), 479–491.

[52] J. Michel and R. Robert, *Large deviations for young measures and statistical mechanics of infinite dimensional dynamical systems with conservation law*, Communications in Mathematical Physics **159** (1994), 195–215.

[53] ———, *Statistical mechanical theory of the great red spot of jupiter*, Journal of Statistical Physics **77** (1994), 645–666.

[54] Jonathan Miller, *Statistical mechanics of euler equations in two dimensions*, Phys. Rev. Lett. **65** (1990), no. 17, 2137–2140.

[55] P. J. Morrison, *Hamiltonian description of the ideal fluid*, Reviews of Modern Physics **70** (1998), 467–521.

[56] C. Mouhot and C. Villani, *On the Landau damping*, arXiv:0904.2760 (2009).

[57] L. Onsager, *Statistical hydrodynamics*, Nuovo Cimento **6** (1949), no. 2 (Suppl.), 249–286.

[58] J. Paret and P. Tabeling, *Intermittency in the two-dimensional inverse cascade of energy: Experimental observations*, Phys. Fluids **10** (1998), 3126–3136.

[59] R. Robert, *Etats d'équilibre statistique pour l'écoulement bidimensionnel d'un fluide parfait*, C. R. Acad. Sci. **1** (1990), 311:575–578.

[60] ———, *A maximum-entropy principle for two-dimensional perfect fluid dynamics*, J. Stat. Phys. **65** (1991), 531–553.

[61] ———, *On the Statistical Mechanics of 2D Euler Equation*, Communications in Mathematical Physics **212** (2000), 245–256.

[62] Raoul Robert and Joël Sommeria, *Relaxation towards a statistical equilibrium state in two-dimensional perfect fluid dynamics*, Phys. Rev. Lett. **69** (1992), no. 19, 2776–2779.

[63] R. T. Rockafellar, *Convex analysis*, reprint: 1997 ed., Princeton University Press, Princeton, NJ, 1970.

[64] R. Salmon, *Lectures on Geophysical Fluid Dynamics*, Oxford University Press, 1998.

[65] D. A. Schecter, D. H. E. Dubin, K. S. Fine, and C. F. Driscoll, *Vortex crystals from 2D Euler flow: Experiment and simulation*, Phys. Fluids **11** (1999), 905–914.

[66] R. A. Smith and T. M. O'Neil, *Nonaxisymmetric thermal equilibria of a cylindrically bounded guiding-center plasma or discrete vortex system*, Phys. Fluids B **2** (1990), 2961–2975.

[67] J. Sommeria, *Experimental study of the two dimensional inverse energy cascade in a square box*, J. Fluid. Mech. **170** (1986), 139–168.

[68] J. Sommeria and R. Robert, *Statistical equilibrium states for two-dimensional flows*, J. Fluid Mech. **229** (1991), 291–310.

[69] H. Spohn, *Large Scale Dynamics of Interacting Particles*, Springer, New-York, 2002.

[70] A. Venaille and F. Bouchet, *Statistical Ensemble Inequivalence and Bicritical Points for Two-Dimensional Flows and Geophysical Flows*, Physical Review Letters **102** (2009), no. 10, 104501–+.

[71] Y. Y. Yamaguchi, *One-dimensional self-gravitating sheet model and Lynden-Bell statistics*, Phys. Rev. E **78** (2008), no. 4, 041114–+.

[72] V. Zeitlin, *Finite-mode analogs of 2D ideal hydrodynamics: Coadjoint orbits and local canonical structure*, Physica D Nonlinear Phenomena **49** (1991), 353–362.